Effects of spacing on dynamics of two coupled Bose-Einstein condensates in two finite traps

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Interaction between two coupled Bose-Einstein condensates (BECs) is investigated by the variational approach in two finite traps, and the effects of the spacing between the two traps on dynamics of the two BECs are analyzed. The spacing determines the stable condition of stationary states, affects the existence condition of each BEC, and changes the switching and self-trapping effects on the two BECs. The dynamic mechanism is demonstrated by performing a coordinate of classical particle moving in an effective potential field, and confirmed by the evolution of the atom population transferring ratio.

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Bose-Einstein condensates (BECs) offer a unique possibility of studying nonlinear effects using matter waves, such as the possibility of four wave mixing, the creation of topological structures, the creation of solitons, as well as other demonstrations of the super-fluid character[1]. A mean-field description for the macroscopic BEC wavefunction is constructed using the Hartree-Fock approximation and results in Gross-Pitaevskii equation (GPE) that supports solitonic solutions. The existence of solitonic solutions is a very general feature of nonlinear wave equations. Depending on the repulsive or attractive nature of the inter-atomic interactions, the GPE allows for either dark or bright soliton, respectively[2].

One of the most important aspects of BECs is that they are unstable against interactions between neighboring BECs. For example, dynamical instabilities, switching and trapping characteristics of coupled BECs have been studied in the framework of almost all nonlinear evolution equations possessing soliton solutions[3−10]. Soliton trains consisting of up to ten solitons have been observed in the experiment, and it is found that the neighboring solitons repel or attract each other with a force that is dependent on their spacing. Physically, this can be understood from the fact that the anti-symmetric nature of the many-soliton wave function prevents the solitons from penetrating each other. Recently, a relevant interesting issue is to learn interaction of different types of condensates, including the coupled BEC solitons in traps. The question then arises how to affect or even guide their interaction. The motion of the coupled condensates results in a train of self-coherent solitonic pulses, and becomes the basis of the pulsed atomic soliton lasers[11].

In this letter, we use a variational approach to investigate the effects of the spacing on the dynamics of the coupled BECs, in which we describe the wave-function of the individual solitons as a Gaussian function, and the width (number of atoms in each BEC) and phase (the effective wave-vector) of the Gaussian function can be varied.

Two coupled BECs in two traps can be described by the following coupled nonlinear Schrödinger equation[5−8]

$$j\frac{\partial u_i}{\partial t}+\frac{1}{2}\frac{\partial^2 u_i}{\partial z^2}+|u_i|^2 u_i=V(z)u_i+Ku_{3-i}, \qquad (1)$$

where $u_i (i=1,2)$ is the condensate wave-function, t the normalized time, K the linear coupling coefficient arising out of overlaps of the transverse parts of the wave-functions, and $V(z)$ the normalized confining potential of each trap in the longitudinal direction (z direction).

Compared with the temporal dependence, the spatial dependence is weak, and the two traps behave independently with the well-known ground state solution in the form of Gaussian-shape (quasi-soliton). Then we adopt trial functions below as the solution to Eq. (1)

$$u_i(z,t)=\frac{N_i}{\sqrt[4]{\pi}}\exp[-\frac{N_i^2}{2}(z\pm\rho/2)^2+jk(z\pm\rho/2)+j\phi_i], \qquad (2)$$

where ρ is the spacing between the two BECs, k the effective wave-vector, and ϕ_i $(i=1,2)$ the local phase. $N_i=\int_{-\infty}^{\infty}|u_i|^2\mathrm{d}z$ $(i=1,2)$ is the number of atoms in each trap, and $N=N_1+N_2$ is the total number of atoms in the two traps (a conserved quantity). In the evolution of the two BECs, the wave-function u_i $(i=1,2)$ retains the Gaussian-shape given by Eq. (2), the effective wave-vector and the number of atoms in each trap become functions of time, but the spacing and the local phase are slowly varying quantities.

In the present study, the two coupled BECs in two finite traps are considered, and the potential of the trap is[12]

$$V(z)=\begin{cases}0 & |z|\le 1\\ V_0 & |z|>1\end{cases}, \qquad (3)$$

where $V(z)$ represents the square well potential of finite trap, and V_0 is the amplitude of the potential. This potential gives analytic solutions, unlike harmonic traps,

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which in one dimension do not give rise to analytic solutions. It has the advantage of having a direct analog in the linear Schrödinger equation, for which the stationary states have been worked out completely.

The averaged Lagrangian of Eqs. (1) can be defined as usual variational approach

$$L(t) = \int_{-\infty}^{\infty}\left\{\sum_{i=1}^{2}\left[\frac{j}{2}\left(u_i^*\frac{\partial u_i}{\partial t} - u_i\frac{\partial u_i^*}{\partial t}\right) - \frac{1}{2}\left|\frac{\partial u_i}{\partial z}\right|^2 + \frac{1}{2}|u_i|^4 - V|u_i|^2\right] - K(u_1^*u_2 + u_2^*u_1)\right\}\mathrm{d}z$$
$$= \sum_{i=1}^{2}\left[\mu\frac{N_i\rho}{2}\frac{\mathrm{d}k}{\mathrm{d}t} + aN_i^3 - bV_0N_i\right] - \frac{2\sqrt{2}KN_1N_2}{\sqrt{N_1^2+N_2^2}}\exp\left[-\frac{N_1^2N_2^2}{2(N_1^2+N_2^2)}\rho^2\right]\cos[k\rho + (\phi_2-\phi_1)], \quad (4)$$

where $a = (\frac{1}{4} - \frac{1}{2\sqrt{2\pi}})$, and $b = \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp[-z^2]\mathrm{d}z - \int_{-1}^{1}\frac{1}{\sqrt{\pi}}\exp[-z^2]\mathrm{d}z = 0.158$.

The equations of motions for the wave-vector and the number of atoms in each trap are obtained from the averaged Lagrangian by using $\mathrm{d}L(t)/\mathrm{d}\sigma - \mathrm{d}[\mathrm{d}L(t)/\mathrm{d}\sigma]/\mathrm{d}t = 0$ $(\sigma = N_1, N_2, k)$, and the two important equations are obtained

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2(1-R^2)\rho\,K}{\sqrt{1+R^2}}\exp\left[-\frac{(1-R^2)^2}{16(1+R^2)}N^2\rho^2\right]\sin(k\rho+\phi),$$
$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{\rho}\left\{-3aN^2R + \frac{2(3+R^2)KR}{(1+R^2)^{3/2}}\exp\left[-\frac{(1-R^2)^2}{16(1+R^2)}N^2\rho^2\right]\cos(k\rho+\phi)\right.$$
$$\left. - \left[\frac{(1-R^2)^2(2-R^2)KR}{16(1+R^2)^{5/2}}\rho^2\right]\exp\left[-\frac{(1-R^2)^2}{16(1+R^2)}N^2\rho^2\right]\cos(k\rho+\phi)\right\}, \quad (5)$$

where $\phi = \phi_2 - \phi_1$ is the local phase difference between the two BECs. $R(t) = (N_2 - N_1)/N$ is the atom population transferring ratio, and $N_1 = N(1-R)/2$ and $N_2 = N(1+R)/2$ are used.

According to the equation $H = \sum_{i=1}^{2}[\partial N_i/\partial t * \partial L/\partial(\dot{N}_i)] + \partial k/\partial t * \partial L/\partial(\dot{k})] - L$, the Hamiltonian of the BEC system is

$$H = \sum_{i=1}^{2}(aN_i^3) + \frac{2\sqrt{2}KN_1N_2}{\sqrt{N_1^2+N_2^2}}\exp\left[-\frac{N_1^2N_2^2}{2(N_1^2+N_2^2)}\rho^2\right]\cos[k\rho+(\phi_2-\phi_1)]$$
$$= \frac{a}{4}N^3(1+3R^2) - bV_0N + \frac{KN(1-R^2)}{\sqrt{1+R^2}}\exp\left[-\frac{(1-R^2)^2}{16(1+R^2)}N^2\rho^2\right]\cos(k\rho+\phi). \quad (6)$$

Equations (5) and (6) determine the dynamics of the two coupled BECs.

The stationary states can be obtained by setting the time derivatives in Eq. (5) to zero. The stationary states in absence of the potential deviation are

$$R = 0 \quad (N_1 = N_2 = N/2),$$
$$\text{for} \quad k = \frac{(n\pi - \phi)}{\rho} \quad (n = 0, \pm1, \pm2, \cdots), \quad (7)$$
$$R = \pm1 \quad (N_1 = 0, \text{ or } N_2 = 0),$$
$$\text{for} \quad k = \frac{1}{\rho}[\arccos(\frac{3\sqrt{2}aN^2}{4K}) - \phi]. \quad (8)$$

The stability issue for the stationary states can be discussed by performing a standard linear stability analysis[5]. Small fluctuations around the stationary solution are introduced, Eq. (5) is linearized. With respect to small perturbations, we find the possibly stable state of stationary states is $R = 0$ for only $k = (2n\pi - \phi)/\rho$ $(n = 0, \pm1, \pm2, \cdots)$. The stability condition against small perturbation is $[-3aN^2 + (6K - 6N^2\rho^2)\exp(-N^2\rho^2/16)] \leq 0$. The stable states of stationary states can be realized if the total number of atoms is larger than $[2(K - N^2\rho^2)\exp(-N^2\rho^2/16)/a]^{1/2}$ or the coupling coefficient is smaller than $[aN^2\exp(N^2\rho^2/16)/2 + N^2\rho^2]$. The spacing plays an important role in motion of the two coupled BECs because the interaction strictly depends on the spacing.

The stationary states corresponding to the disappearance ($R = \pm1$, namely $N_1 = 0$ or $N_2 = 0$) of either BEC can be seen in Eq. (8), which shows that the disappearance can be avoided if $\cos(k\rho + \phi) = \frac{3\sqrt{2}aN^2}{4K} > 1$. The disappearance can be realized if the total number of atoms is larger than $[4K/3\sqrt{2}a]^{1/2}$ or the coupling coefficient is smaller than $[3\sqrt{2}aN^2/4]$.

The effect mechanism is demonstrated by performing a coordinate of classical particle moving in an effective potential field. Linearizing Eq. (5) in R only, Eq. (5) reduces into the very simple form as below

$$\frac{\mathrm{d}^2k}{\mathrm{d}t^2} = -6aKN^2\exp\left[-\frac{N^2\rho^2}{16}\right]\sin(k\rho+\phi)$$
$$+K^2(6 - 6N^2\rho^2)\exp\left[-\frac{N^2\rho^2}{8}\right]\sin(2k\rho+2\phi). \quad (9)$$

The effective interaction potential between the two BECs is given by

$$V_{\mathrm{eff}}(k) = -\frac{1}{\rho}\left\{6aN^2K\exp\left[-\frac{N^2\rho^2}{16}\right]\cos(k\rho+\phi)\right.$$
$$\left. -K^2(3 - \frac{\rho^2}{2})\exp\left[-\frac{N^2\rho^2}{8}\right]\cos(2k\rho+2\phi)\right\}. \quad (10)$$

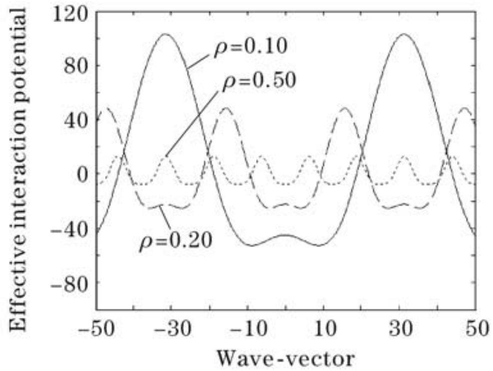

Fig. 1. Effective interaction potential versus the wave-vector.

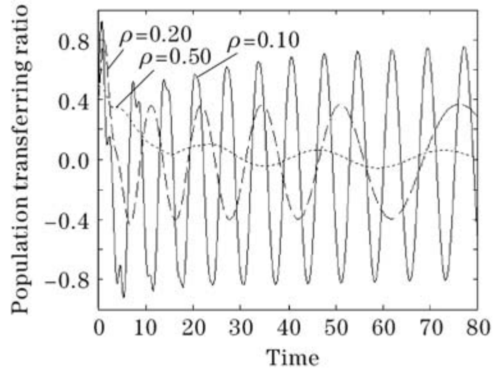

Fig. 2. Atom population transferring ratio versus time.

This suggests a mechanical analogy in which particles move in the effective interaction potential relating to the spatial coordinate k. The effective interaction potential versus the wave-vector is shown in Fig. 1, and the spacings are selected as $\rho = 0.10$, 0.20, and 0.50. The system parameters are that the number of the total atoms is $N = 5$, the local phase difference is $\phi = 2\pi$ and the coupling coefficient is $K = 1.0$. We can see that the effective potential $V_{\rm eff}(k)$ has some small valleys around $k = 2n\pi/\rho$ $(n = 0, \pm 1, \pm 2, \cdots)$, where the particle can oscillate, and the effective interaction potential depends strictly on the spacing. For example, the number of valleys becomes few, and the depth of each valley increases as the spacing becomes small. On the other hand, the depth of each valley decreases as the spacing becomes large. The effective interaction potential becomes nearly zero as the spacing is large enough, and this means that the interaction of the two BECs disappears. The physical mechanism implies that the two BECs can be regarded as the two repellent particles, whose moves may be confined in the lattices (relating to the spatial coordinate k) of the effective interaction potential, which may affect the moving dynamics of the particles. For example, when the spacing is small, the effective interaction potential is large, their moves are confined in the lattices due to the strong interaction. When the spacing becomes large, the potential becomes small, and their moves are released partly due to the reduction of the potential. When the spacing becomes large enough, the potential becomes very small, and their moves are free from the interaction. These features illuminate that the self-trapping and switching dynamics of the two coupled BECs are sensitive to the change of the spacing. For example, the stationary states can be reset by the change of the spacing because their effective wave-vectors are given by $k = (2n\pi - \phi)/\rho$ $(n = 0, \pm 1, \pm 2, \cdots)$.

To understand the effect of the spacing on the switching and self-trapping as well as the dynamics of the two BECs, we can investigate the atom population transfer between the two BECs as the time changes. In Fig. 2, the atom population transferring ratio versus time is plotted by numerically solving Eq. (5) with a fourth order variable-step Runge-Kutta algorithm. The given conditions are the same as those in Fig. 1, and the initial population transferring ratio is $R = 0.5$. The selected spacings are given by $\rho = 0.10$, 0.20, and 0.50. As seen clearly in the figure, the spacing plays an important role in the evolution of the population transferring ratio. For example, the transferring ratio nearly periodically oscillates around $R = 0$ in different amplitudes because of the different spacings, and the oscillation period becomes long as the spacing is large. Disappearance of either BEC does not take place as time changes, and the reason is that the avoidance condition of disappearance is satisfied. If there is small spacing ($\rho = 0.10$), the transferring ratio oscillates in periodical manner with large amplitude, and the atom population transferring procedures are of near-zero average population imbalance. If there is the spacing of $\rho = 0.20$, the transferring ratio oscillates in small amplitude, and the population transferring procedures are of zero average population imbalance. If there is the large spacing of $\rho = 0.50$, the transferring ratio oscillates in very small amplitude because of the very large spacing and the weak interaction, and the oscillation period becomes very long.

In summary, the interaction between the two coupled BECs are investigated by the variational approach in two finite potentials, and the effects of the spacing between the two traps on dynamics of the two coupled BECs are analyzed. It is shown that the spacing determines the stable condition of the stationary states, affects the existence condition of each BEC, and changes the switching and self-trapping effect on the two coupled BECs. The dynamic mechanism is demonstrated by performing a coordinate of classical particle moving in an effective interaction potential, and confirmed by the evolution of the atom population transferring ratio versus time. For example, the atom population transfer between the two BECs depends strictly on the spacing. When the spacing is small, the atom population transferring ratio periodically oscillates in large amplitude, and the interaction between the two BECs is working. When the spacing becomes large, the atom population transferring ratio periodically oscillates in very small amplitude, and the interaction dies down. These results remind that the behaviors of the two BECs would be sensitive to the change of the spacing, and the interaction related to the spacing is one of the most important aspects of the pulsed atomic soliton lasers.

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