Fractional Fourier transform of apertured paraboloid refracting system

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Received July 13, 2006

The limitation of paraxial condition of paraboloid refracting system in performing fractional Fourier transform acts like an aperture, which makes the system different from ideal systems. With aperture expanded as the sum of finite complex Gaussian terms, a more practical approximate analytical solution of fractional Fourier transform of Gaussian beam in an apertured paraboloid refracting system is obtained and also numerical investigation is presented. Complicated and practical fractional Fourier transform systems can be constructed by cascading several apertured paraboloid refracting systems which are the simplest and the most basic units for performing more precise transform.

OCIS codes: 070.2590, 050.1220, 120.4820, 140.3300.

Physically, optical fractional Fourier transform is performed in laterally finite system, however, mathematically, fractional Fourier transform is defined in laterally infinite system[1−7]. This results in the difference between mathematical anticipation and physical transformation. Generally, three aspects contribute to this difference. Firstly, the Fresnel diffraction in free space is based on paraxial approximation[8]; secondly, the phase transform function of lens is derived from the assumption that omitting the primary spherical aberration factor or the higher one is permissible; finally, the real system is always an apertured system. Among three factors, which one is dominant or decisive depends on the specific problem. The factor that sets the minimum beam width to the diffraction wave is usually dominant. In most cases, the condition of omitting the spherical aberration in deriving the phase transform function sets the minimum beam width to fractional Fourier transform of input wave. It is also a paraxial condition. For paraboloid refracting system, it has no primary spherical aberration factor[9], but for thin paraboloid refracting system, it also demands the paraxial condition. With this in mind, we take this limitation as a virtual aperture. The paraxial approximation of Fresnel diffraction in free space is further taken into account, the virtual aperture on the paraboloid refracting surface can be mapped to the virtual aperture on the input plane, so fractional Fourier transform integral is confined to this virtual aperture.

The fractional Fourier transform implemented by apertured systems has been extensively studied[10−14], but the apertured paraboloid refracting system is the simplest and the most basic system for performing fractional Fourier transform. By cascading two or more different apertured paraboloid refracting systems, the complicated, practical, and precise fractional Fourier transform systems can be constructed. Thus, investigating the apertured paraboloid refracting system is meaningful and significant.

Figure 1 is a paraboloid refracting system which is capable of implementing fractional Fourier transform of pth order when $z_1 = \frac{nF_1 \sin\varphi}{1-\cos\varphi}$, $z_2 = \frac{n_0 F_1 \sin\varphi}{1-\cos\varphi}$, and $r = \frac{(n_0-n)F_1}{\sin\varphi}$, where $\varphi = p\frac{\pi}{2}$. The focal distance of thin refracting lens located between plane P_1 and plane P_2 is $f' = \frac{n_0 F_1}{\sin\varphi}$. Here "thin" means that the thickness between plane P_1 and plane P_2 is much less than the radius r at the vertex of paraboloid refracting surface. This assumption, in turn, requires that the beam width $s_1 s_4$ of incident wave on the thin refracting lens is limited. Taking this limitation as a virtual aperture with radius b, the aperture function $A(x_1)$ is

$$A(x_1) = \begin{cases} 0 & |x_1| > b \\ 1 & |x_1| \le b \end{cases}. \tag{1}$$

To the thin paraboloid refracting lens, the phase transform function is[9]

$$t(x_1) = \exp[-jk(n_0 - n)\frac{x_1^2}{2r}], \tag{2}$$

where the constant phase factor has been omitted. So

$$u'(x_1) = A(x_1)t(x_1)u(x_1). \tag{3}$$

We can expand the aperture function $A(x_1)$ into a finite sum of complex Gaussian functions[13−15]

$$A(x_1) = \sum_{m=1}^{M} A_m \exp(-\frac{B_m}{b^2}x_1^2), \tag{4}$$

where A_m and B_m denote the expansion and the Gaussian coefficients, respectively, which could be obtained by optimization computation directly. Table 1 lists a set of this two coefficients when m varies from 1 to 10, A_m

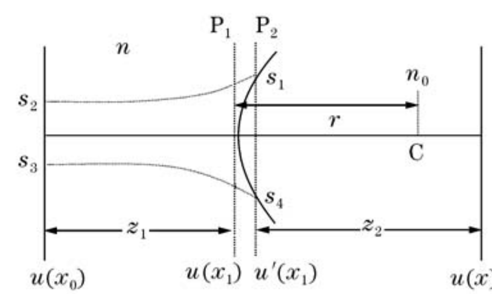

Fig. 1. Fractional Fourier transform system constructed with a single paraboloid refracting surface. The input function is an apertured Gaussian beam with a waist at input plane.

Table 1. Expansion Coefficients of Hard-Edged Aperture

m	A_m	B_m
1	$11.428 + 0.95175j$	$4.0697 + 0.22726j$
2	$0.06002 - 0.08013j$	$1.1531 - 20.933j$
3	$-4.2743 - 8.5562j$	$4.4608 + 5.1268j$
4	$1.6576 + 2.7015j$	$4.3521 + 14.997j$
5	$-5.0418 + 3.2488j$	$4.5443 + 10.003j$
6	$1.1227 - 0.68854j$	$3.8478 + 20.078j$
7	$-1.01016 - 0.26955j$	$2.5280 - 10.310j$
8	$-2.5974 + 3.2202j$	$3.3197 - 4.8008j$
9	$-0.14840 - 0.31193j$	$1.9002 - 15.820j$
10	$-0.20850 - 0.23851j$	$2.6340 + 25.009j$

and B_m are both complex constants.

Let $A_m = R_m + T_m j$, $B_m = L_m + O_m j$, it is convenient for following development of integral. Then aperture function can be written as

$$A(x_1) = \sum_{m=1}^{10} (R_m + T_m j) \exp(-\frac{L_m + O_m j}{b^2} x_1^2). \quad (5)$$

Figure 2 is the real part and imaginary part of this aperture function. Figure 3 is the amplitude and phase of this Gaussian expansion. In Figs. 2 and 3, the radius of this aperture $b = 10$ mm. From the figure, we can see that the expansion formula is only the approximate simulation of the real aperture. Usually, the higher the

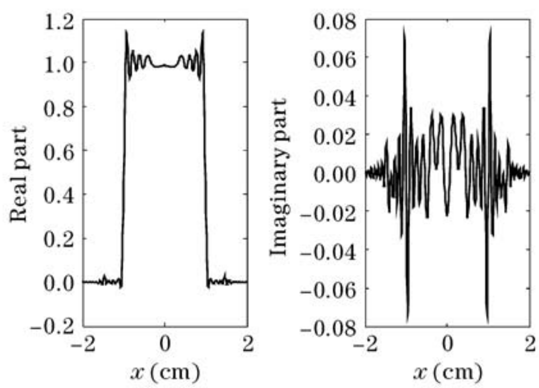

Fig. 2. Real and imaginary parts of complex Gaussian expansion, $M = 10$, $b = 10$ mm.

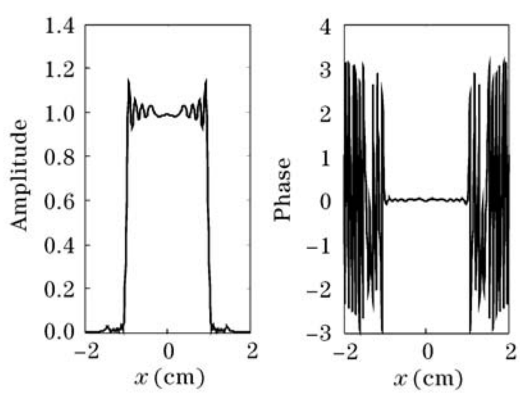

Fig. 3. Amplitude and phase of complex Gaussian expansion, $M = 10$, $b = 10$ mm.

M is, the more precise the simulation is.

The relations among $u(x_0)$, $u(x_1)$, $u'(x_1)$, and $u(x)$ as illustrated in Fig. 1 are

$$u(x_1) = \frac{\exp(jknz_1)}{j\frac{\lambda}{n}z_1} \int_{-\infty}^{\infty} u(x_0) \exp\{j\frac{kn}{2z_1}[(x_1 - x_0)^2]\} \mathrm{d}x_0, \quad (6)$$

$$u(x) = \frac{\exp(jkn_0 z_2)}{j\frac{\lambda}{n_0}z_2} \int_{-\infty}^{\infty} u'(x_1) \exp\{j\frac{kn_0}{2z_2}[(x - x_1)^2]\} \mathrm{d}x_1, \quad (7)$$

where the integral limits are all infinite as in the integral of mathematical definition of fractional Fourier transform. However, as discussed above, the thin lens is only effective within a laterally finite range.

To the Gaussian beam, two virtual apertures have different diameters which are related with location of beam waist and diverging angle. As shown in Fig. 1, let the waist of Gaussian beam at input plane, the complex amplitude distribution at P_1 plane can be written as

$$u(x_1) = u_0 \frac{\omega_0}{\omega(z_1)} \exp[-\frac{x_1^2}{\omega^2(z_1)}] \exp[-j\frac{kn_0 x_1^2}{2h(z_1)}], \quad (8)$$

where

$$h(z_1) = n_0(z_1 + \frac{\pi^2 \omega_0^4}{z_1 \lambda^2}), \quad (9)$$

$$\omega(z_1) = \sqrt{\omega_0^2 + (\frac{\lambda z_1}{\pi \omega_0})^2}, \quad (10)$$

λ is the wavelength of the Gaussian beam, ω_0 is the waist width. Inserting Eqs. (2), (5), and (8) into Eqs. (3), we can obtain

$$u'(x_1) = u_0 \frac{\omega_0}{\omega(z_1)} \exp[-\frac{x_1^2}{\omega^2(z_1)}] \exp[-j\frac{kx_1^2}{2h(z_1)}]$$
$$\times \sum_{m=1}^{M} (R_m + T_m j) \exp(-\frac{L_m + O_m j}{b^2} x_1^2) \exp(-j\frac{kn_0}{2f'} x_1^2), \quad (11)$$

where $f' = \frac{n_0 r}{n_0 - n}$. Further, by inserting Eq. (11) into Eq. (7), we get

$$u(x) = u_0 \frac{\omega_0}{\omega(z_1)} \frac{\exp(jkn_0 z_2)}{j\frac{\lambda}{n_0}z_2} \sum_{m=1}^{M} (R_m + T_m j)$$
$$\times \int_{-\infty}^{\infty} \exp[-\frac{x_1^2}{\omega^2(z_1)}] \exp[-j\frac{kx_1^2}{2h(z_1)}] \exp(-j\frac{kn_0}{2f'} x_1^2)$$
$$\times \exp(-\frac{L_m + O_m j}{b^2} x_1^2) \exp[j\frac{kn_0}{2z_2}(x - x_1)^2] \mathrm{d}x_1. \quad (12)$$

Using following integral formula

$$\int_{-\infty}^{\infty} \exp(-l^2 x_1^2 - q x_1) \mathrm{d}x_1 = \exp(\frac{q^2}{4l^2}) \frac{\sqrt{\pi}}{l} \quad [\mathrm{Re} l^2 > 0], \tag{13}$$

we obtain

$$u(x) = u_0 \frac{\omega_0}{\omega(z_1)} \frac{\exp(jkn_0 z_2)}{j\frac{\lambda}{n_0} z_2} \sum_{m=1}^{M} (R_m + T_m j)$$

$$\times \exp(j\frac{kn_0}{2z_2} x^2) \exp[\frac{-(\frac{kn_0 x}{z_2})^2}{U_m + V_m j}]$$

$$\times \frac{\sqrt{\pi}}{\frac{S_m}{\sqrt{2(\sqrt{S_m^2+Q_m^2}-Q_m)}} + \sqrt{\frac{\sqrt{S_m^2+Q_m^2}}{2} - \frac{Q_m}{2} j}}, \tag{14}$$

where

$$U_m = \frac{4L_m}{b^2} + \frac{4}{\omega^2(z_1)}, \tag{15}$$

$$V_m = \frac{4O_m}{b^2} + 2kn_0[\frac{1}{f'} + \frac{1}{h(z_1)} - \frac{1}{z_2}], \tag{16}$$

$$S_m = \frac{O_m}{b^2} + kn_0[\frac{1}{2f'} + \frac{1}{2h(z_1)} - \frac{1}{2z_2}], \tag{17}$$

$$Q_m = \frac{L_m}{b^2} + \frac{1}{\omega^2(z_1)}. \tag{18}$$

For numerical simulation of fractional Fourier transform of Gaussian beam in hard-apertured paraboloid refracting system, the order of fractional Fourier transform is $p = 0.6$, $\varphi = 0.3\pi$, $F_1 = 200$ mm, $n = 1$, $n_0 = 1.4$, $r = 98.88$ mm, $z_1 = 392.72$ mm, $z_2 = 549.81$ mm, $f' = 346.11$ mm, the waist width of Gaussian beam at the input plane is $\omega_0 = 10$ mm, the beam width at the aperture plane $\omega(z_1) \cong 10$ mm, $h(z_1) = 8.786e + 8$ mm. Figure 4 is the amplitude distribution of fractional Fourier transform of Gaussian beam when the aperture size of the paraboloid refracting system is decreased from much bigger than the beam width at the aperture plane to much less than the beam width. It should be noted that the amplitude distribution of fractional Fourier transform of Gaussian beam in a circular hard-apertured system vibrates around two points corresponding to two edges of aperture along x_1 axis. When the aperture size tends to small one, the vibrations extends to the center of the transformed Gaussian beam and correspondingly the beam profile is distorted more seriously. This performance may be originated from aperture edge diffraction and approximation simulation of aperture complex Gaussian expansion. As shown in Figs. 2 and 3, the real part and amplitude of complex Gaussian expansion also vibrate at aperture edges. It is anticipated that, with the increase of expansion series terms and coefficients precision, the edge-effect will decrease.

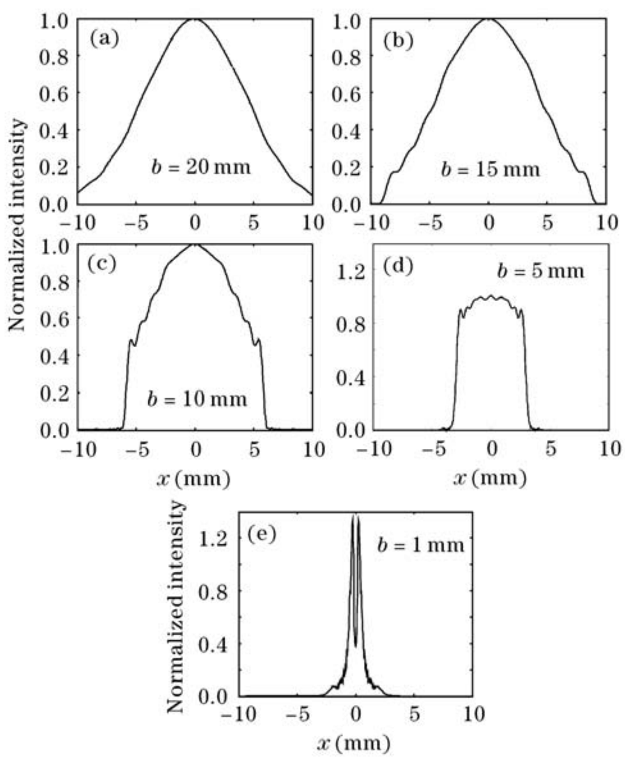

Fig. 4. Distortion process of amplitude distribution of fractional Fourier transform of Gaussian beam when the paraboloid refracting system is apertured.

In conclusion, by expanding aperture function as the sum of finite complex Gaussian terms, the more practical and precise fractional Fourier transform made by the apertured paraboloid refracting system is analytically and numerically studied. The result could be useful in configuring the more complicated and precise fractional Fourier transform systems.

This work was supported by Ludong University Research Foundation under Grant No. 20042811. J. Chen's e-mail address is jnchen1963@yahoo.com.cn.

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