Study on the approximative formula for the far-field of a Gaussian beam under circular aperture diffraction and its divergence

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The approximative formula for the far-field diffraction of a Gaussian beam through a circular aperture is obtained by using the superposition of Gaussian beams instead of the aperture function, and the explicit expression for calculating the beam divergence is also gained. Using the formula, the influences of the aberrations on the far-field wavefront and the beam's divergence are researched, and the results show that the large aberrations badly affect the far-field wavefront and the divergence. It is suggested that the aberrations and the diffractions should be avoided when designing the transmitter.

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The semiconductor laser diode communication system has been studied for many years[1−3]. It is very interesting and will be extensively applied in the future. We are doing the relative research in this field[4−10]. We have designed a transmitter for breadboard model. The optical beam out of the transmitter propagates over 45000 km, so the beam wavefronts are required to have high quality and the divergence must be reached the diffraction-limited. But the actual wavefronts are badly affected by the aberrations and the diffraction of the emitter aperture.

When the Gaussian beam emits through a circular aperture, the beam is diffracted. Now we assume that the normal amplitude of the Gaussian beam is

$$a(r_0) = \exp(-\frac{r_0^2}{\omega_0^2}), \tag{1}$$

where ω_0 is the waist size of the Gaussian beam. Let the added focused phase aberration is represented by $\mathrm{err}(r_0)$. According the Ref. [11]

$$\begin{aligned}\mathrm{err}(r_0) &= \exp(-j2\pi\frac{\Delta\omega}{\lambda}) \\ &= \exp(-j2\pi\frac{\Delta\omega_\mathrm{m}}{\lambda})\exp(j2\pi\frac{\Delta\omega_\mathrm{m} r_0^2}{\lambda R_0^2}),\end{aligned} \tag{2}$$

where R_0 notes the radius of the circular aperture and $\Delta\omega_\mathrm{m}$ represents the maximum of the focused aberration. Let the aperture function is represented by $\mathrm{circ}(r_0)$, then the amplitude of the aberration Gaussian beam after the aperture is expressed as

$$\begin{aligned}b(r_0) &= a(r_0)\mathrm{err}(r_0)\mathrm{circ}(r_0) \\ &= \exp(-\frac{r_0^2}{\omega_0^2})\exp(-j2\pi\frac{\Delta\omega_\mathrm{m}}{\lambda})\exp(j2\pi\frac{\Delta\omega_\mathrm{m} r_0^2}{\lambda R_0^2})\mathrm{circ}(r_0).\end{aligned} \tag{3}$$

Using the diffraction theory, the wavefront of optical far-field can be expressed as

$$\begin{aligned}A(r,z) &= \frac{2\pi}{j\lambda z}\exp(j\frac{2\pi}{\lambda}z) \\ &\times \int_0^\infty \exp(j\frac{2\pi}{\lambda}\frac{r^2+r_0^2}{2z})J_0(\frac{2\pi}{\lambda}\frac{rr_0}{z})b(r_0)r_0\mathrm{d}r_0 \\ &= \frac{k}{jz}C\int_0^R \exp(jk\frac{r^2+r_0^2}{2z})\exp(-\frac{r_0^2}{\omega_0^2}) \\ &\times \exp(\frac{jk\Delta\omega_\mathrm{m} r_0^2}{R_0^2})J_0(k\frac{rr_0}{z})\mathrm{circ}(r_0)r_0\mathrm{d}r_0,\end{aligned} \tag{4}$$

where $C = \exp(jkz)\exp(-jk\Delta\omega_\mathrm{m})$, $k = \frac{2\pi}{\lambda}$, and J_0 is the zero order Bessel function.

Equation (4) is changed into non-dimension expression as

$$\begin{aligned}A(\tau,\eta) &= \frac{2C}{j\eta}\int_0^1 \exp(j\frac{\tau^2+\zeta^2}{\eta})\exp(-\frac{R_0^2}{\omega_0^2}\zeta^2) \\ &\times \exp(jk\Delta\omega_\mathrm{m}\zeta^2)J_0(2\frac{\tau\zeta}{\eta})\mathrm{circ}(\zeta)\zeta\mathrm{d}\zeta \\ &= \frac{2C}{j\eta}\int_0^1 \exp(j\frac{\tau^2+\zeta^2}{\eta})\exp(-B_0\zeta^2) \\ &\times J_0(2\frac{\tau\zeta}{\eta})\mathrm{circ}(\zeta)\zeta\mathrm{d}\zeta,\end{aligned} \tag{5}$$

here, $B_0 = (R_0^2/\omega_0^2) - jk\Delta\omega_\mathrm{m}$, $\tau = r/R_0$, $\zeta = r_0/R_0$, $\eta = z/L$. $L = kR_0^2/2$ notes the Rayleigh distance of the circular aperture with radius R_0. Equation (5) represents the far-field wavefront of the Gaussian beam out of the circular aperture. It is very difficult to get the analytic expression of the far-field wavefront. In order to research the far-field, it is requested to carry the numerical integration of Eq. (5). It is also difficult to obtain the analytic expression of the far-field. To solve this difficulty, we will expand the circular function $\mathrm{circ}(\zeta)$ according to complex Gaussian function[12], and then the analytic approximate expression can be obtained.

The circular aperture

$$\mathrm{circ}(\zeta) = \begin{cases} 1 & 0 \le \zeta < 1 \\ 0 & \zeta > 0 \end{cases} . \tag{6}$$

It can be expressed as the lineal superposition of the complex Gauss function, then[12]

$$\mathrm{circ}(\zeta) = \sum_{n=1}^{10} A_n \exp(-B_n \zeta^2), \quad \zeta \in [0, \infty). \tag{7}$$

The corresponding A_n, B_n can be got by optimizing through computer, and they have be given in Table 1[8]. The beam through the circular aperture is expressed as

$$b(\zeta) = \begin{cases} \exp(-B_0 \zeta^2) & 0 \le \zeta < 1 \\ 0 & \zeta > 1 \end{cases} . \tag{8}$$

Using Eq. (7), $b(\zeta)$ can be changed into

$$b(\zeta) = \exp(-B_0\zeta^2)\mathrm{circ}(\zeta)$$

$$= \exp(-B_0\zeta^2) \times \sum_{n=1}^{10} A_n \exp(-B_n\zeta^2), \quad \zeta \in [0, \infty). \tag{9}$$

Through the transformation, the upper limit of the integration is extended into infinite, and then Eq. (5) can be expressed by

$$A(\tau, \eta) = \frac{2C}{j\eta} \int_0^1 \exp(j\frac{\tau^2 + \zeta^2}{\eta}) \exp(-B_0\zeta^2) \times J_0(2\frac{\tau\zeta}{\eta}) \mathrm{circ}(\zeta)\zeta \mathrm{d}\zeta$$

$$= \frac{2C}{j\eta} \int_0^\infty \exp(j\frac{\tau^2 + \zeta^2}{\eta}) J_0(2\frac{\tau\zeta}{\eta}) \exp(-B_0\zeta^2) \times \sum_{n=1}^{10} A_n \exp(-B_n\zeta^2)\zeta \mathrm{d}\zeta$$

$$= \frac{2C}{j\eta} \int_0^\infty \exp(j\frac{\tau^2 + \zeta^2}{\eta}) J_0(2\frac{\tau\zeta}{\eta}) \times \sum_{n=1}^{10} A_n \exp(-B'_n\zeta^2)\zeta \mathrm{d}\zeta, \tag{10}$$

here $B'_n = B_0 + B_n$. Using $\int_0^\infty J_0(\alpha t) \exp(-\gamma^2 t^2) t\mathrm{d}t = \frac{1}{2}\gamma^{-2}\exp(-\frac{1}{4}\gamma^{-2}\alpha^2)$, the Eq. (10) is changed into

$$A(\tau, \eta) = \frac{2C}{j\eta} \int_0^\infty \exp(j\frac{\tau^2 + \zeta^2}{\eta}) J_0(2\frac{\tau\zeta}{\eta}) \times \sum_{n=1}^{10} A_n \exp(-B'_n\zeta^2)\zeta \mathrm{d}\zeta$$

$$= C \sum_{n=1}^{10} A_n \frac{\exp[-B'_n\tau^2/(1 + jB'_n\eta)]}{(1 + jB'_n\eta)}. \tag{11}$$

Equation (11) is one of the main results, and it is the analytic approximate expression for the far-field wavefront. Whether Eq. (11) can approximate the Eq. (5) very well? We will perform the numerical calculation of Eqs. (5) and (11), respectively, and then compare them with each other.

Using the Eq. (11), the far-field divergence can be conveniently got, and it is one of purposes to obtain the Eq. (11). Here the far-field divergence θ is defined the ratio of the radius r_e, it corresponds to the point where the value of amplitude of the optical field is $1/e$ of its maximum and the propagation distance is z. Only knowing the normalized distance τ_e which corresponds to r_e, the far-field divergence can be obtained

$$D = \frac{|A(0, \eta)|}{e} = \left| \frac{C}{e} \sum_{n=1}^{10} \frac{A_n}{(1 + jB'_n\eta)} \right|, \tag{12}$$

$$|A(\tau_e, \eta)| = \left| C \sum_{n=1}^{10} A_n \frac{\exp[-B'_n\tau_e^2/(1 + jB'_n\eta)]}{(1 + jB'_n\eta)} \right| = D, \tag{13}$$

$$\theta = \tau_e/\eta. \tag{14}$$

Let the radius of the circular pupil $R_0 = 10$ mm, the beam's wavelength $\lambda = 0.8$ μm, the Rayleigh distance

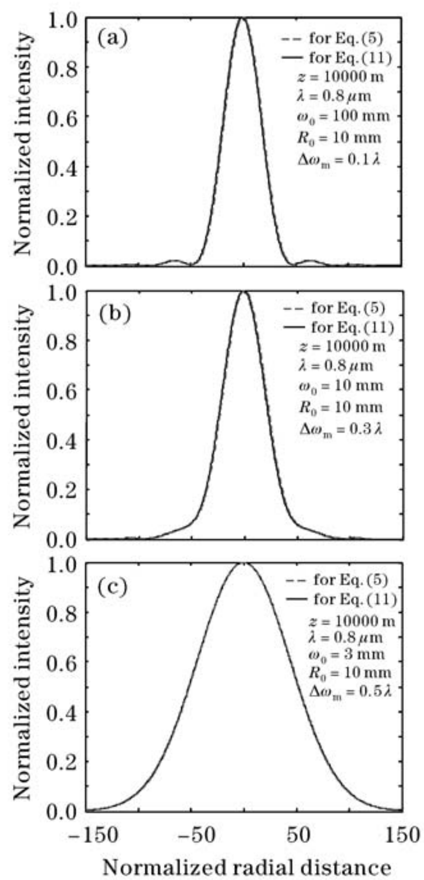

Fig. 1. Normalized intensity distributions in the radial direction of Eqs. (5) and (11) for different parameters in the same transmission distance.

$L = kR_0^2/2 = 393$ m and the far-field distance $z = 10000$ m $\gg L$. Changing the beam waist size ω_0 and the maximum of the aberration $\Delta\omega_{\rm m}$, we discuss three kinds of representative conditions, which correspond to $\omega_0 \gg R_0$, $\omega_0 \approx R_0$, $\omega_0 < R_0$, respectively, and the corresponding results are shown in Figs. 1(a)—(c).

From Fig. 1(a), when $\omega_0 = 100$ mm $\gg R_0 = 10$ mm, $\Delta\omega_{\rm m} = 0.1\lambda$, the beam can be looked as parallel wave, and the results from Eqs. (5) and (11) are completely same to each other under $z = 10000$ m $\gg L$. Corresponding Fig. 1(b) $\omega_0 = 10$ mm $= R_0$, $\Delta\omega_{\rm m} = 0.3\lambda$, although there is large defocus aberration, Eqs. (5) and (11) are completely same to each other again. For the third condition $\omega_0 = 3$ mm $< R_0 = 10$ mm, and $R_0/\omega_0 \approx 3.3$, this indicates that the beam can pass through the circular pupil freely, even $\Delta\omega_{\rm m} = 0.5\lambda$, the results for Eqs. (5) and (11) are still complete superposition. The above three results under different conditions indicate that whether the beam has weakly or strongly diffraction or even freely propagate, the analytic approximation represented by Eq. (11) has high accuracy compared with the Eq. (5) for the Gaussian beam under a given propagation distance and defocus aberration.

Whether the results from Eqs. (11) and (5) are same for different propagation distances? We have compared them (see in Figs. 2(a)—(c)). The circular radius and the maximum value of the defocus aberration have the same values as above, and let $\Delta\omega_{\rm m} = 0.2\lambda$, ω_0 are 100, 10 and 3 mm, respectively, and the corresponding z are 3000, 6000 and 9000 m, respectively. From Figs. 2(a)—(c), even for different propagation distances and for weak or strong diffraction, Eqs. (11) and (5) still can be superposition.

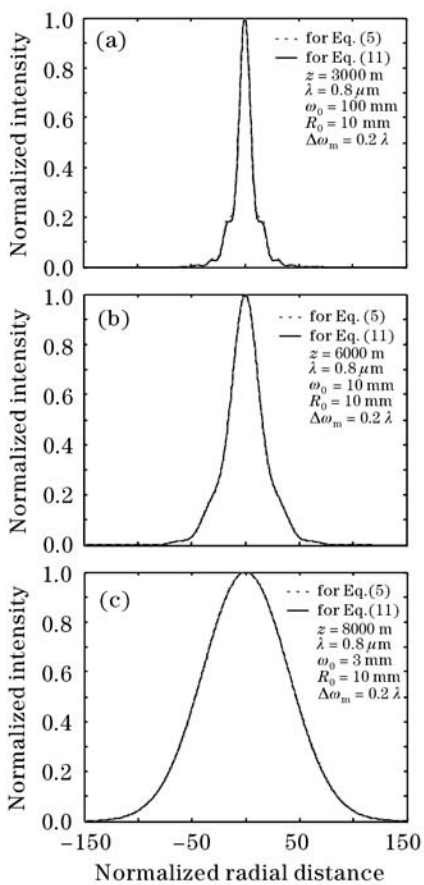

Fig. 2. Normalized intensity distributions in the radial direction of Eqs. (5) and (11) for different parameters in the different transmission distances.

We also compared the result using Eq. (5) to calculate the beam divergence with using Eq. (11), as shown in Fig. 3. From it, we can find that the divergence using Eq. (5) accords well with that using Eq. (5), Eq. (11) is reasonable from the other hand.

Synthesizing the above different conditions, it is reasonable to use Eq. (11) representing the far-field diffraction of the Gaussian beam with defocus aberration.

Equation (11) is reasonable and is of great benefit to analyze the far-field wavefront of the truncated Gaussian beams under circular aperture diffraction, the influence of aberration on the far-field wavefront and the beam divergence are analyzed as follows.

The far-field wavefront changes with the aberration as shown in Fig. 4. In order to contrast, the parallel beam diffracted under circular aperture is given too, which corresponds to the Airy disk. From it, we can find that the far-field wavefront is little affected by the aberration when $\Delta\omega_{\rm m}$ is under 0.2λ; if $\Delta\omega_{\rm m} > 0.2\lambda$, the far-field wavefront is changed remarkably with increasing $\Delta\omega_{\rm m}$, even the concave wavefront of the far-field appears, and which shows that the aberration can badly affect the far-field wavefront and some steps to reduce the aberration are necessary.

The far-field divergence is key parameter of the transmitter. It changes with the different waist sizes ω_0, the

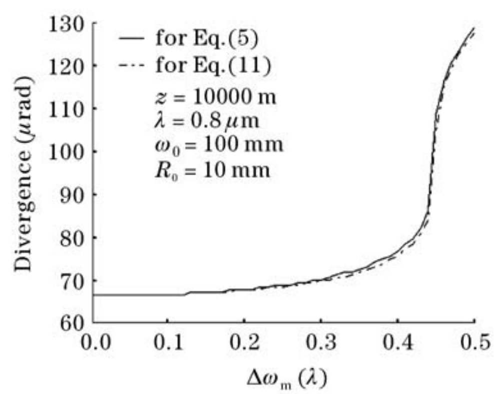

Fig. 3. Comparing the divergence using Eq. (5) with using Eq. (11).

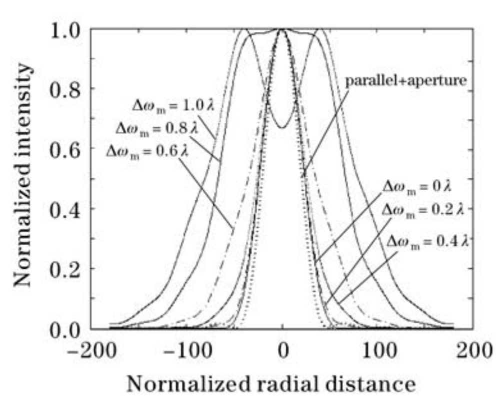

Fig. 4. Far-field wavefront changes with the aberrations.

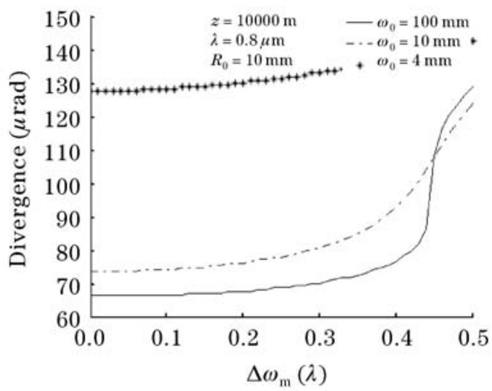

Fig. 5. Beam's divergence changes with the aberrations.

different pupil radii r_0, and the different phase aberrations. For a designed transmitter, ω_0 and R_0 are given, but the aberration is variable, which affects the divergence as shown in Fig. 5. As an example, just the defocused phase aberration is considered, and this is based on that other aberrations of the transmitter have already been minimized. Here, $R_0 = 10$ mm. Let $\omega_0 = 100$, 10, 4 mm and change $\Delta\omega_{\mathrm{m}}$ from zero to 0.5λ. From the figure, we can see that the divergence is increased monotonously with $\Delta\omega_{\mathrm{m}}$; on the other hand, whether there is diffraction or not, the divergence will be changed with changing $\Delta\omega_{\mathrm{m}}$. We also can find that the diffraction is stronger, the divergence is changed much faster. In order to preserve the beam quality, we should reduce the diffraction and eliminate the aberrations.

In this paper, using the complex Gaussian decomposition, the analytic approximate expression for the far-field wavefront of the truncated Gaussian beam under circular aperture diffraction is gained. According to this approximate expression, the far-field divergence of the truncated Gaussian beam is obtained. The influences of the aberrations on the far-field wavefront and the beam divergence are researched, the results show that the large aberrations badly affect the far-field wavefront and the divergence. These give us some useful suggestions that the aberrations and the diffractions should be avoided when designing the transmitter, and this work can be benefit to some engineering application too.

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