Geometric phase of spin-1 in a rotating magnetic field

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Received December 27, 2005

The quantum phase and the related fields have attracted considerable attention. The geometric phase of spin-1/2 particle in the magnetic field has been discussed comprehensively, but few of spin-1. In this paper, the exact solution of spin-1 was obtained by using rotational frame method. The problems of the Rabi oscillation, dynamical phase, and geometric phase were solved.

OCIS codes: 000.3860, 000.6800, 020.5580.

Since the geometric phase was introduced by Berry[1], it has been researched widely, and its existence has been verified experimentally in many physical branches, such as nuclear physics, atomic and molecular physics, optics and the solid physics. Simultaneously, there are many developments on research method[2−6] .

There have already been comprehensive discussions for the geometric phase of spin-1/2 particle in the conic magnetic field[7,8]. However, only a few articles penetrate with discussing the geometric phase of spin-1 particles. In 1989, Mizrahi[9] brought forward the invariable method of constructing general Hamiltonian, and obtained the Aharonov-Anandan phase of spin-1/2 particle in the invariable magnetic field. In his method, the most important thing is to construct an invariable operator $\hat{I}(t)$. $\hat{I}(t)$ is simple and the solving process is easy for spin-1/2 system, however, it is complicated and the solving process is difficult for spin-1 system. Cui *et al.*[10] have treated the Berry phase of spin-1 particles in a conic magnetic field by solving the Schrödinger equation directly, whereas their discussions are not extended to other complex circumstances. In this article, the exact solution of spin-1 particle has been obtained in the rotating magnetic field by using rotating frame method. We have obtained the exact analytic expressions of the polarization vector, three-level Rabi oscillation, geometric phase, and dynamic phase.

First of all, we discuss the motion of a spin moment $\hat{\mathbf{M}}_{\mathrm{s}}$ in a rotating magnetic field $\mathbf{B}(t) = B_0\mathbf{e}_z + B_1(\cos\omega t\mathbf{e}_x + \sin\omega t\mathbf{e}_y)$. And the Hamiltonian of this system can be expressed as

$$\hat{H}(t) = -\hat{\mathbf{M}}_{\mathbf{s}}\cdot\mathbf{B}(t) = \gamma\hat{\mathbf{J}}_{\mathbf{s}}\cdot\mathbf{B}(t)$$

$$= \hbar(\Omega_0\hat{I}_z + \Omega_1\hat{I}_x\cos\omega t + \Omega_1\hat{I}_y\sin\omega t), \quad (1)$$

where $\hat{\mathbf{J}}_{\mathbf{s}}$ is the spin momentum, ω is the rotational frequency of magnetic field, $\gamma = g_{\mathrm{s}}\frac{e}{2M}$ is the particle gyromagnetic ratio. And $\hat{\mathbf{I}} = \hat{\mathbf{J}}_{\mathbf{s}}/\hbar$, $\Omega_0 = \gamma B_0$, $\Omega_1 = \gamma B_1$.

We use the rotating frame method to get the solution of the spin-1 particle. At first, a rotational transform $\hat{R}(t)$ is introduced, which transfers wave function $\psi(t)$ to the wave function $\psi_{\mathrm{e}}(t)$ of rotating frame $\psi(t) = \hat{R}(t)\psi_{\mathrm{e}}(t) = \exp(-i\omega\hat{I}_z t)\psi_{\mathrm{e}}(t)$. Then we get the Schrödinger equation in a rotational frame as

$$i\frac{\partial\psi_{\mathrm{e}}(t)}{\partial t} = \{(\Omega_0-\omega)\hat{I}_z + \Omega_1\hat{I}_x\}\psi_{\mathrm{e}}(t).$$

It is noted that the Hamiltonian is time-independent in rotational frame. So it is easy to get the equation

$$\psi_{\mathrm{e}}(t) = \exp[-i\{(\Omega_0-\omega)\hat{I}_z + \Omega_1\hat{I}_x\}t]\psi_{\mathrm{e}}(0)$$

$$= \exp\{-i\Omega(\mathbf{n}\cdot\hat{\mathbf{I}})t\}\psi_{\mathrm{e}}(0), \quad (2)$$

where $\mathbf{n}(\frac{\Omega_1}{\Omega}, 0, -\frac{\Delta}{\Omega}) = \mathbf{n}(\sin\alpha, 0, \cos\alpha)$.

The exact solution can be expressed as

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi_{\mathrm{e}}(0)\rangle\,, \quad (3)$$

here $U(t) = \exp(-i\omega\hat{I}_z t)\exp[-i\Omega(\mathbf{n}\cdot\hat{\mathbf{I}})t]$ is a unitary matrix, which can be expressed as a 3×3 matrix with the following elements,

$$U_{11} = U_{33}^* = \mathrm{e}^{-i\omega t}[\frac{1}{2}\sin^2\alpha$$

$$+\frac{1}{2}(1+\cos\alpha)\cos\Omega t + i\cos\alpha\sin\Omega t],$$

$$U_{12} = \mathrm{e}^{-i\omega t}U_{21} = -U_{32}^* = -\mathrm{e}^{-i\omega t}U_{23}^*$$

$$= \mathrm{e}^{-i\omega t}[-\frac{1}{2\sqrt{2}}\sin 2\alpha(1-\cos\Omega t) + \frac{i}{\sqrt{2}}\sin\alpha\sin\Omega t],$$

$$U_{13} = U_{31}^* = \mathrm{e}^{-i\omega t}[-\frac{1}{2}\sin^2\alpha(1-\cos\Omega t)],$$

$$U_{22} = \cos^2\alpha + \sin^2\alpha\cos\Omega t.$$

Equation (3) is the final result obtained by using rotating frame method.

Next, we compute some important physical quantities, such as the polarization, Rabi oscillation, dynamical phase, and geometric phase by using the result solved above.

Now we calculate the spin polarization vector and the Rabi oscillation. From Eq. (3) we get

$$\mathbf{P}(t) = \langle\psi(t)|\,\hat{\mathbf{I}}\,|\psi(t)\rangle$$

$$= \langle\psi(0)|\,\hat{U}^{+}\hat{\mathbf{I}}\hat{U}\,|\psi(0)\rangle = tr[\hat{\rho}_0\hat{U}^{+}\hat{\mathbf{I}}\hat{U}], \quad (4)$$

where $\rho_0 = |\psi(0)\rangle\langle\psi(0)|$ is the initial density matrix. Substituting Eq. (3) into Eq. (4), we obtain a compact result

$$\mathbf{P}(t) = K(t)\mathbf{P}(0),$$

$$K(t)=\begin{pmatrix}\cos\omega t & \sin\omega t & 0\\ \sin\omega t & -\cos\omega t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos^2\alpha\cos\Omega t+\sin^2\alpha & -\cos\alpha\sin\Omega t & \sin 2\alpha(1-\cos\Omega t)/2\\ -\cos\alpha\sin\Omega t & -\cos\Omega t & \sin\Omega t\\ \sin 2\alpha(1-\cos\Omega t)/2 & \sin\alpha\sin\Omega t & \cos^2\alpha+\sin^2\alpha\cos\Omega t\end{pmatrix}, \quad (5)$$

$\alpha=\arcsin(\Omega_1/\Omega)$.

Now we discuss the Rabi oscillation of above three-level system. Let $W_\pm(t)=|C_\pm(t)|^2$ and $W_0(t)=|C_0(t)|^2$ represent the probabilities of $\hat{J}_z$ taking eigenvalues $\pm\hbar$, 0 respectively under the state $|\psi(t)\rangle=(C_+(t),C_0(t),C_-(t))^T$. Figure 1 shows the oscillation of $W_i(t)$ with a period $T=\frac{2\pi}{\Omega}$ which is half of the $\mathbf{P}(t)$'s period $\frac{2\pi}{\omega}$.

In the following, we compute the non-adiabatic dynamic phase and geometric phase with the Aharonov-Anandan formula[11]. Undergoing a cyclic evolution, the total phase $\phi(T)$ of the system is equal to the sum of geometric phase $\gamma_{\mathrm{g}}(T)$ and dynamic phase $\gamma_{\mathrm{d}}(T)$,

$$\phi(T)=\gamma_{\mathrm{d}}(T)+\gamma_{\mathrm{g}}(T),$$

and

$$\left.\begin{aligned}\gamma_{\mathrm{d}}(\tau)&=-\frac{1}{\hbar}\int_0^\tau dt\,\langle\psi(t)|\hat{H}(t)\,|\psi(t)\rangle\\ \gamma_{\mathrm{g}}(\tau)&=i\int_0^\tau\left\langle\tilde{\psi}(t)\right|\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle\mathrm{d}t\end{aligned}\right\}, \quad (6)$$

here $\tilde{\psi}(t)=\exp(-i\omega t)\psi(t)$.

Next, we compute the geometric phase and dynamic phase. Taking the initial state as eigenstate of $J_z=\hbar$,

$$|\psi_+(0)\rangle=\left(\frac{1+\cos\vartheta_0}{2}\mathrm{e}^{-i\varphi_0},\sqrt{\frac{1}{2}}\sin\vartheta_0,\frac{1-\cos\vartheta_0}{2}\mathrm{e}^{i\varphi_0}\right)^{\mathrm{T}},$$

T denotes the transposition. By a direct computation we obtain

$$\begin{aligned}\gamma_{\mathrm{d}}(\tau)&=-\frac{1}{\hbar}\int_0^\tau\mathrm{d}t\,\langle\psi(0)|\hat{H}_U(t)\,|\psi(0)\rangle\\ &=-\frac{\sin\Omega\tau}{\Omega}[\cos\vartheta_0(\Omega_0\sin^2\alpha-\frac{\Omega_1}{2}\sin 2\alpha)\\ &\quad+\sin\vartheta_0\cos\varphi_0(\Omega_1\cos^2\alpha-\frac{\Omega_0}{2}\sin 2\alpha)],\end{aligned}$$

where $\hat{H}_U(t)=\hat{U}^\dagger\hat{H}(t)\hat{U}$. With the formula we get the geometric phase as

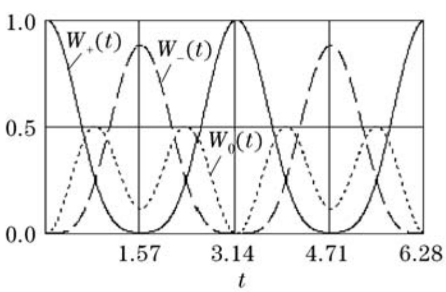

Fig. 1. Rabi oscillation in rotating magnetic field for spin-1 particles. $\frac{\Omega_0}{\omega}=1.5$, $\frac{\Omega}{\omega}=2$, $\vartheta_0=\varphi_0=0$.

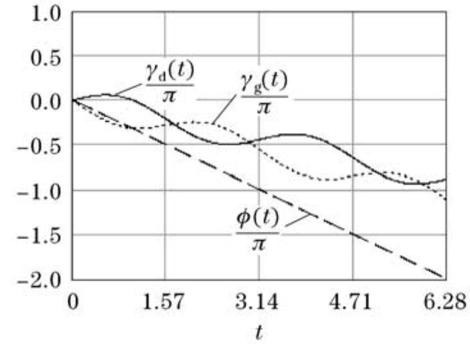

Fig. 2. Dynamic phase and geometric phase within one period of initiation to be the eigenstate $\hbar$. $\omega=1$, $\Omega_0=1.5$, $\Omega=2$, $\vartheta_0=\varphi_0=0$.

$$\begin{aligned}\gamma_{\mathrm{g}}(\tau)&=i\int_0^\tau\left\langle\tilde{\psi}(0)\right|U^\dagger\dot{U}\left|\tilde{\psi}(0)\right\rangle\mathrm{d}t\\ &=\frac{\sin\Omega\tau}{\Omega}(\omega\sin^2\alpha\cos\vartheta_0-\frac{\omega}{2}\sin 2\alpha\sin\vartheta_0\cos\varphi_0)\\ &\quad-\frac{1-\cos\Omega\tau}{\Omega}(\omega\sin\vartheta_0\sin\alpha\sin\varphi_0)\\ &\quad+\tau(\cos\vartheta_0\cos\alpha+\sin\vartheta_0\cos\varphi_0\sin\alpha)(\omega\cos\alpha+\Omega)-\omega\tau.\end{aligned}$$

The total phase $\phi(\tau)=\gamma_{\mathrm{g}}(\tau)+\gamma_{\mathrm{d}}(\tau)=-\omega\tau$ is acquired within one period as expected. Figure 2 shows the change of phases with time.

As taking the initial state to be the eigenstate of $J=0$, $|\psi_0\rangle=\left(-\sqrt{\frac{1}{2}}\sin\vartheta_0\mathrm{e}^{-i\varphi_0},\cos\vartheta_0,\sqrt{\frac{1}{2}}\sin\vartheta_0\mathrm{e}^{i\varphi_0}\right)^{\mathrm{T}}$, after a similar computation mentioned above, we obtain $\gamma_{\mathrm{d}}=0$, $\gamma_{\mathrm{g}}=-\omega\tau$, and $\phi=-\omega\tau$. On the other hand, if the initial state is the eigenstate of $J=-\hbar$, $|\psi_-\rangle=\left(\frac{1}{2}(1-\cos\vartheta_0)\mathrm{e}^{-i\varphi_0},-\sqrt{\frac{1}{2}}\sin\vartheta_0,\frac{1}{2}(1+\cos\vartheta_0)\mathrm{e}^{i\varphi_0}\right)^{\mathrm{T}}$, we get $\gamma_{\mathrm{g}}^-(\tau)=-\gamma_{\mathrm{g}}^+(\tau)-2\omega\tau$, $\gamma_{\mathrm{d}}^-(\tau)=-\gamma_{\mathrm{d}}^+(\tau)$, $\phi^-(\tau)=\phi^+(\tau)$, where the superscripts $\pm$ stand for the eigenvalues $\pm\hbar$.

We remark that the system of spin-1 particle can be seen as spin parallel triplet of two spin-1/2 particles on neglecting interaction (such as double-electron). Also atom, molecule or nucleus system of total angular momentum $J=\hbar$ can gain A-A geometric phase in rotating field[12]. This results may be used for some three-level systems and some optic systems. Song *et al.*[13] studied the fifth-order attosecond sum-frequency polarization beat (FASPB) in a cascade three-level system. In some problems of quantum dot, for example, if the potential field is syntonic and only reserving the lowest three-level energy and neglecting other higher-level energy under low temperature (leakage approximation), this simplified three-level energy system (such as three-level system of spin-1 composed with two protons[14]) can be described as spin-1.

As another possible application of this work, we note that the geometric quantum computation has attracted many attentions owing to its high fidelity[15−21]. Recently, using an orthogonal method, to two-level system, Gao *et al.*[20,21] presented a scenario for the non-adiabatic geometric quantum computation, the quantum gates are obtained with all advantages of holonomic geometric quantum gates. We remark that the scheme can be parallel extended to the three-level system investigated in this paper.

In summary, this paper obtain the exact solution of magnetic moment of spin-1 particle in the rotating magnetic field by the rotating frame method. We also compute various important physical quantities, such as three-level Rabi oscillation, geometric phase etc..

This work was supported by the National Natural Science Foundation of China under Grant No. 10204008, 10474022, and the Key Project of Chinese Ministry of Education (No. 205113). X. Zhang's e-mail address is zhangxy-2002@163.com. L. Hu's e-mail address is hulian0822@163.com.

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