Transformation of general astigmatic Gaussian beams in a four-dimensional phase space

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A phase space model of two-dimensional (2D) Gaussian beam propagation is generalized for three-dimensional (3D) general astigmatic Gaussian beam passing through first-order optical system. The general astigmatic Gaussian beam is represented by a four-dimensional (4D) phase super-ellipsoid that defined by an associated 4×4 real matrix, then the transformation formula of the phase super-ellipsoid of the beam through first-order optical system is derived. In particular, in the phase space framework, the beam propagation factor M^2 value is proved to be a ratio of phase area of real beam to ideal beam, and a novel approach for a qualitative examination of the properties of fractional Fourier transform (FRT) for the beam is also provided.

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The phase space model for propagation of two-dimensional (2D) Gaussian beams has been proposed some time ago[1,2]. According to this model, the propagation of Gaussian beams is represented by a transformation of the phase ellipse associated a positive real symmetric 2×2 beam matrix in the phase plane. However, the actual case of light beam propagation is a three-dimensional (3D) problem (two transversal dimensions x, y, and one axial dimension z) and most of the Gaussian beams exhibit partially coherent and astigmatic characteristics. The Collett-Wolf source, for instance, is typical source that emits partially coherent Gaussian-Shell model (GSM) beams[3]. GSM beams and later twisted GSM beams have attracted particular interest because such beams not only can be analyzed theoretically[4−13] but can also be constructed in the laboratory[14]. In the theoretical aspect, the Wigner distribution function is widely used to treat the propagation and imaging of both twisted and non-twisted GSM beams[7−11]. Therefore, in the general case, 2D phase space model for transmission of Gaussian beam or simple astigmatic Gaussian beam can be generalized in 3D to fully characterize the most general astigmatic beam (coherent or incoherent) propagation with ten physical parameters.

In this letter, we propose four-dimensional (4D) phase space transformation for general astigmatic beam (coherent or incoherent) propagating through a first-order optical system by employing the Liouville theorem in optics derived from the Fermat's principle[1]. The general astigmatic (or anisotropic) beam is fully characterized by a 4×4 real symmetric matrix σ, which may be called the generalized phase space beam matrix. Then the transformation law of σ through a first-order optical system is derived. It is shown that σ-transformation recovers the second-moments transformation law by using the method of the Wigner distribution. In particular, within the framework of phase space picture, the beam propagation factor M^2 value gives a measure of "how many times of phase area of the ideal beam" is the phase area of the real beam in each phase space corresponding to each transverse direction, also a novel approach for a qualitative examination of the properties of fractional Fourier transform (FRT) for the general astigmatic Gaussian beam is provided.

We consider a paraxial general astigmatic Gaussian beam traveling along z axis, denoted with x and y, the small displacements in the two directions transverse to the beam direction at distance along the optic axis. From the statistical non-wave viewpoint, propagation of a light ray, like the motion of a particle in classical mechanics, can be fully depicted by its Hamilton canonical equation. Naturally, a general astigmatic Gaussian beam consisting of a number of rays similar to a group of particles in classical statistics can also be represented by a phase space volume. The position of a light ray and its "momentum" can construct a 4D phase space in which a ray is represented by a point, this phase representative point moves in the phase space as the light ray travels in real space guided by Hamilton canonical equation. A general astigmatic Gaussian beam with limited width and slope range occupies a limited volume in the phase space, the volume moves with varying shape in phase space while the beam propagates through a first-order optical system. Following the evolution of the volume in phase space we get the information about the beam in real space. Liouville theorem states that the volume in the phase space (X, P) is conserved during the beam propagation. Here, $X = [x\ y]$, $P = [n\mathrm{d}x/\mathrm{d}z\ n\mathrm{d}y/\mathrm{d}z]$[1] and n is the refractive index of the material where the beam propagating. For simplicity, we assume that $n = 1$ in the following.

In order to extend the formalism of 2D phase ellipse in the (x, x') phase plane to the 3D case, we must define a 4D phase super-ellipsoid in the (X, X') phase space as

$$[X\ X']\sigma^{-1}[X\ X']^T = 1, \quad (1)$$

where $X' = [x'\ y']$ and $x' = \mathrm{d}x/\mathrm{d}z$, $y' = \mathrm{d}y/\mathrm{d}z$, the superscript T means transposition, σ defines the generalized phase space beam matrix of general astigmatic Gaussian beam which can be written as

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{bmatrix},$$

$$\sigma_1 = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} \sigma_{xx'} & \sigma_{xy'} \\ \sigma_{yx'} & \sigma_{yy'} \end{bmatrix},$$

$$\sigma_3 = \begin{bmatrix} \sigma_{x'x} & \sigma_{x'y} \\ \sigma_{y'x} & \sigma_{y'y} \end{bmatrix}, \ \sigma_4 = \begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix}, \quad (2)$$

where, by symmetry consideration, $\sigma_1 = {\sigma_1}^T$, $\sigma_3 = {\sigma_2}^T$ and $\sigma_4 = {\sigma_4}^T$, σ_1 is a transverse spot width matrix and σ_4 is a divergence angle matrix, while σ_2 is a coupling matrix. Generally, σ_1 and σ_4 contain three real parameters, respectively, and σ_2 contains four real parameters, therefore, σ contains ten independent real parameters. It is clear that σ is a positive real symmetric matrix. All the known families of Gaussian beams are subsets of the beam described by Eq. (2). (a) If $\sigma_{xy'} \neq 0$ and $\sigma_{yx'} \neq 0$, and these three matrices σ_1, σ_2, and σ_4 have different principal axes (the axes along which the matrices are diagonal), the beam is a general astigmatic beam; (b) If $\sigma_{xy'} = \sigma_{yx'} = 0$, and the two matrices σ_1 and σ_4 have the same principal axes and are aligned (or rotated) with respect to the laboratory axes x and y, the beam is an aligned simple astigmatic (rotated simple astigmatic) beam; and (c) If $\sigma_{xy'} = \sigma_{yx'} = 0$, and the two matrices σ_1 and σ_4 are proportional to the identity matrix, the beam is stigmatic and has rotational symmetry.

At the conventional level of ray optics, a first-order optical system changes the ray parameters by the simple transformation[15]

$$[X \ X']_{\rm o}^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} [X \ X']_{\rm i}^T, \quad (3)$$

where the subindices 'o' and 'i' variables refer to the output and input planes, respectively, and $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the ray-transfer matrix that must satisfy

$$\det S = 1, \quad (4)$$

A, B, C, and D are all 2×2 sub-matrices of the ray-transfer matrix S.

We further analyze the beam behavior in the phase space framework. Since $[X \ X']_{\rm i}^T$ and $[X \ X']_{\rm o}^T$ denote the phase representative point vectors corresponding to the input and output planes, respectively, Eq. (3) relates the output phase representative point vector to the input one and the $ABCD$ sub-matrices. From Eq. (1), the phase super-ellipsoid corresponding to the input plane is

$$[X \ X']_{\rm i} \sigma_{\rm i}^{-1} [X \ X']_{\rm i}^T = 1. \quad (5)$$

Equation (5) is equivalent to

$$[X \ X']_{\rm i} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \right)^{-1}$$

$$\times \sigma_{\rm i}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} [X \ X']_{\rm i}^T = 1,$$

i.e.,

$$\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix} [X \ X']_{\rm i}^T \right]^T \left[\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sigma_{\rm i} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \right]^{-1}$$

$$\times \left[\begin{bmatrix} A & B \\ C & D \end{bmatrix} [X \ X']_{\rm i}^T \right] = 1.$$

With Eq. (3), We may write the phase super-ellipsoid corresponding to the output plane as

$$[X \ X']_{\rm o} \sigma_{\rm o}^{-1} [X \ X']_{\rm o}^T = 1, \quad (6)$$

therefore

$$\sigma_{\rm o} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sigma_{\rm i} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = S \sigma_{\rm i} S^T, \quad (7)$$

which is the generalized phase space beam matrix transformation law of the beam, where $\sigma_{\rm i}$ and $\sigma_{\rm o}$ represent the phase space beam matrices corresponding to the input and output planes, respectively. The effect of a first-order system is to transfer the beam matrix $\sigma_{\rm i}$ into beam matrix $\sigma_{\rm o}$ via $\sigma_{\rm o} = S\sigma_{\rm i} S^T$ in terms of phase space model. Since $\sigma_{\rm i}$ is positive real symmetric matrix, so is also $\sigma_{\rm o}$. For the case of GSM beam, we recall that Wigner function is nothing but the phase representative points density function of the phase space (X, X'), it is easy to see that σ is the 4×4 real symmetric variance matrix by using second-moments method. As a consequence we have $\sigma_{\rm W,o} = S\sigma_{\rm W,i} S^T$ from Eq. (7) (where the subindex 'W' stands for the variance matrix of second-moments of Wigner distribution), which is consistent with that of second-moments theory[4−11]. But we think that the phase space description technique is a more direct approach because the essentials of the problem under this consideration are much clearer. Once the σ-matrix is calculated along with optical system, all the information of the beam propagation can be obtained.

We emphasize that some additional information can be obtained from beam matrix σ. Firstly, let's begin with the explanation and analysis of 2D Gaussian beam (one transversal dimension x, and one axial dimension z, we call it ideal beam). Typically, the beam behavior for the phase plane (x, x') can be described by a 2D phase ellipse that defined by an associated 2×2 real matrix

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'x'} \end{bmatrix}. \quad (8)$$

Liouville's theorem states that the area enclosing the representative points of the beam in this phase plane remains constant

$$A_x = A_{p,x} = \pi (\det\sigma)^{1/2} = \pi \sqrt{\sigma_{xx}\sigma_{x'x'} - \sigma_{xx'}^2}$$

$$= \pi \sqrt{\sigma_{0xx}\sigma_{0x'x'}} = \lambda, \quad (9)$$

where λ is the wavelength of Gaussian beam, $A_{p,x}$ is phase area of ideal beam, σ_{0xx} and $\sigma_{0x'x'}$ represent the beam waist width and the divergence angle in the far field, respectively. On the other hand, the beam propagation factor M^2 is

$$M^2 = \frac{\pi}{\lambda}\sqrt{\sigma_{0xx}\sigma_{0x'x'}} = \frac{A_x}{A_{p,x}} = 1, \tag{10}$$

or

$$\sigma_{xx}\sigma_{x'x'} - {\sigma_{xx'}}^2 = \left(\frac{\lambda}{\pi}M^2\right)^2. \tag{11}$$

Remarkably, we notice, from Eq. (10), that the beam propagation factor M^2 characterizes the ratio of phase area of Gaussian beam to that of ideal beam, this means that the phase area of Gaussian beam is the minimum value.

Now, we concentrate on the explanation and analysis of a 3D general astigmatic Gaussian beam that defined by Eq. (2), the width and the divergence angle become generalized parameters that defined by 2×2 matrices, such as σ_1 and σ_4, the extension of the formalism of Eq. (11) to the 3D case requires the definition of a generalized beam propagation factor matrix as

$$\sigma_1\sigma_4 - {\sigma_2}^2 = \left(\frac{\lambda}{\pi}M^2\right)^2, \tag{12}$$

$$M^2 = \begin{bmatrix} M_{xx}^2 & M_{xy}^2 \\ M_{xy}^2 & M_{yy}^2 \end{bmatrix}. \tag{13}$$

The volume of the 4D phase super-ellipsoid, according to Liouville, is an invariant, it follows

$$A = \frac{\pi^2}{2}\sqrt{\det\sigma} = C. \tag{14}$$

Thus, the overall beam propagation factor is defined by

$$M_{\text{eff}}^4 = \frac{A}{A_p} = \frac{\sqrt{\det\sigma}}{\sqrt{\det\sigma_p}} = (\frac{\pi}{\lambda})^2\sqrt{\det\sigma}, \tag{15}$$

where σ_p is generalized beam matrix of ideal beam corresponding to real general astigmatic Gaussian beam given by

$$\sigma_p = \begin{bmatrix} \sigma_{p1} & \sigma_{p2} \\ \sigma_{p3} & \sigma_{p4} \end{bmatrix},$$

$$\sigma_{p1} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{bmatrix}, \ \sigma_{p2} = \begin{bmatrix} \sigma_{xx'} & 0 \\ 0 & \sigma_{yy'} \end{bmatrix},$$

$$\sigma_{p3} = \begin{bmatrix} \sigma_{xx'} & 0 \\ 0 & \sigma_{yy'} \end{bmatrix}, \ \sigma_{p4} = \begin{bmatrix} \sigma_{x'x'} & 0 \\ 0 & \sigma_{y'y'} \end{bmatrix}, \tag{16}$$

with the assumption of symmetric conditions $\sigma_{xx} = \sigma_{yy}$, $\sigma_{x'x'} = \sigma_{y'y'}$ and $\sigma_{xx'} = \sigma_{yy'}$. Moreover, it is easy to see, from Eqs. (7) and (12), that $\text{Tr}M^4$ is also an invariant, which characterizes the beam intrinsic astigmatism given by

$$J = (\frac{\pi}{\lambda})^2\Big[(\sigma_{xx}\sigma_{x'x'} - \sigma_{xx'}^2) + (\sigma_{yy}\sigma_{y'y'} - \sigma_{yy'}^2) + 2(\sigma_{xy}\sigma_{x'y'} - \sigma_{xy'}\sigma_{yx'})\Big]. \tag{17}$$

Consequently, we have

$$J = M_{xx}^4 + M_{yy}^4 + \frac{2\pi^2}{\lambda^2}(\sigma_{xy}\sigma_{x'y'} - \sigma_{xy'}\sigma_{yx'}). \tag{18}$$

It is clear that, in general, M_{xx}^2 and M_{yy}^2 are not invariants of projection. This is in agreement with Liouville's theorem that the area enclosing the representative points in the (x, x') phase plane is not conserved during the beam propagation, so is also in the (y, y') phase plane. These results support that M^2 has provided a figure for comparing different types beams with respect to the ideal beam (the Gaussian beam) and show that M^2 value is a ratio of phase area of real beam to that of ideal beam from the statistical non-wave viewpoint.

In addition, the above picture of phase space analysis allows us to investigate qualitatively the properties of the FRT[16,17] of general astigmatic Gaussian beams. For simplicity, we consider 2D elliptical Gaussian beam (EGB) through optical system for performing the FRT (one-lens system or two-lens system). The propagation of EGB associated with two spatial dimensions is independent, thus we can study in each dimension of the 2D phase space. According to phase space formulation Eq. (7), the effect of FRT optical system is to transfer phase ellipse σ_{i} into σ_{o}, i.e., the beam is represented by an ellipse in each phase plane corresponding to each transverse direction in the output plane. By rotating phase plane coordinate system with angle φ, we can find that the dependence of the beam width on φ is periodic, and the period is π, so is also the beam divergence angle. So we can conclude that the dependence of beam width and beam divergence angle of the EGB on the fractional order p of the FRT are periodic, with a period of 2.

In conclusion, general astigmatic Gaussian beam can be represented by a 4D phase super-ellipsoid that defined by an associated real 4×4 matrix in terms of phase space model, and the transformation formula of the phase super-ellipsoid of the beam passing through a first-order optical system is derived, this formula recovers the second-moments transformation law by using the method of the Wigner distribution. More specifically, from this point of phase space view, the beam propagation factor M^2 value is proved to be a ratio of phase area of real beam to that of ideal beam, also a novel approach for a qualitative examination of the properties of FRT for the beam is provided. Finally, we stress that this geometrical scenario does not offer any advantage in terms of computational efficiency. Apart from its undeniable simplicity and beauty, its benefits lie in gaining insights into the qualitative behavior of the beam evolution.

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