# Revisit to the symmetry relations in diffusely backscattered polarization patterns of turbid media

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As there exists an inconsistency in claiming the symmetrical relations in the 16 Mueller matrix elements used to describe a turbid medium, the author restudies the symmetrical relationships between diffusely backscattered polarization patterns in isotropic turbid media and simulates all two-dimensional elements of diffusely backscattered Mueller matrix in both cases of Rayleigh and Mie scatterings using the double-scattering approximation and the Monte Carlo algorithm, respectively. The previous experimental observations are compared with the numerically determined matrix elements, showing a good agreement in both double-scattering model and Monte Carlo simulation. The symmetrical relations between the Mueller matrix elements are clarified.

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There has been an increasing interest in the study of propagation of light in randomly scattering media, especially for biomedical applications[1−4]. A number of studies have shown that by shining a polarized light beam onto a scattering sample and then analyzing the state of polarization of the diffusely backscattered light, information on the properties of a turbid medium can be obtained[2,4]. In regard to the practical implications, polarization techniques are believed to give a simplified approach for optical imaging of turbid medium compared with time-resolved methods and, in the meantime, to provide additional information about the structures of tissue[5]. However the propagation of polarized light in randomly scattering media is a complex process. A good understanding of this process is essential in order to improve the polarization-based techniques.

It is widely recognized that Stokes vectors and Mueller matrix can provide a complete description of polarized light and optical turbid media. Both theoretical analysis and experimental studies have been carried out. Hielscher *et al.*[6] used a Stokes vector/Mueller matrix approach to describe polarized light scattering in order to achieve a full experimental characterization of the optical properties of a sample under investigation. In a recent theoretical study, Ambirajan *et al.*[7] used a Monte Carlo technique to study the degree of polarization of the diffusely backscattered light emerging from a turbid media. Rakovic *et al.*[8,9] and Bartel *et al.*[10] developed Monte Carlo algorithms to study the backscattered intensity patterns and compared their simulation results with the experimental data. All of the above studies were conducted on the isotropic turbid media without the consideration of the birefringence effect on polarization. Most recently, Wang *et al.*[11−13] used a time-resolved Monte Carlo technique to characterize the propagation of polarized light in both homogeneous turbid medium and linearly birefringent turbid media. Notwithstanding, there exists an inconsistency in these studies in claiming the symmetrical relations in the 16 Mueller matrix elements: the experimental results from Hielsher *et al.* apparently showed that the off-diagonal backscattered Mueller matrix elements are all symetric[6,10]; however, other studies[8,11,14], predominately through theoretical studies, showed that only some of off-diagonal backscattered Mueller matrix elements are symmetric, whereas the others are anti-symmetric.

In order to develop a rigorous and accurate method in both theoretical analysis and experimental studies, the controversy regarding to symmetrical relations between the Mueller matrix elements needs to be clarified. Therefore in the present study the theoretical treatment of the Mueller matrix representation of the diffusely backscattered patterns with the aid of the double scattering approach[14] and Monte Carlo simulation technique are to be revisited, when a Mueller matrix of the turbid medium is written as

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}.$$

It will be shown that in the case of Rayleigh scattering, the symmetrical relations between off-diagonal backscattered Mueller matrix elements exist; and in the case of Mie scattering, the off-diagonal elements in the last row ($m_{14}$, $m_{24}$, $m_{34}$) and last column ($m_{41}$, $m_{42}$, $m_{43}$) are anti-symmetrical or otherwise almost zero, whereas the other off-diagonal matrix elements are symmetrical, based on the restudied analytical calculation and Monte Carlo simulation results, which are in good agreement with the published experimental results.

Rakovic and Kattawar demonstrated a relatively simple analytical double-scattering model[14], which revealed that double scattered light could be used to predict qualitatively the polarization patterns of diffusely backscattered light emerging from turbid media. From its general formula derived with this technique, for the effective backscattered Mueller matrix elements (in the case of Mie scattering), some of off-diagonal elements are symmetrical (such as, $m_{12}$ and $m_{21}$, $m_{24}$ and $m_{42}$), whereas the others are anti-symmetrical (such as, $m_{13}$ and $m_{31}$,

$m_{23}$ and $m_{32}$, $m_{34}$ and $m_{43}$). However, from the previous experimental observations conducted by Hielscher *et al.*[6,10], it was noticed that all the off-diagonal elements are symmetrical, which means $m_{ij} = m_{ji}$.

To study this difference, the double scattering model[14] is first reconsidered. Let $\mathbf{S}_0 = [I_0, Q_0, U_0, V_0]^{\mathrm{T}}$ be the Stokes vector that corresponds to the irradiance of incident light with respect to the $x$-$z$ plane (reference plane) with $z$ being the incident beam direction. By Ref. [14], under the assumption that the scattering of light is incoherent and the contribution to the backscattered light that comes from the multiply scattered photons is small, after double scattering, in which, after the second scattering, photons exist in the medium, which is assumed to occupy the lower half space, $z \leq 0$, the Stokes vectors of the backscattered light, $S_2^{\mathrm{bs}'}(\rho, \phi)$, in the scattering plane can be written as

$$S_2^{\mathrm{bs}'}(\rho, \phi) = L_2(\rho) R(-\phi) S_0,$$

$$L_2(\rho) = \mu_{\mathrm{s}}^2 \int_{-\infty}^{0} \int_{-\infty}^{0} \frac{\mathrm{d}z \mathrm{d}z'}{r^2}$$

$$\times \{\exp[-\mu_{\mathrm{t}}(|z| + |z'| + r)] M(\pi - \theta) M(\theta)\}, \quad (1)$$

where $r = [\rho^2 + (z - z')^2]^{1/2}$, $\tan\theta = \rho/(z - z')$, $\rho$ is the distance on the surface of the media between incoming beam and backscattering light, $\mu_{\mathrm{t}}$ and $\mu_{\mathrm{s}}$ are the extinction and scattering coefficients, respectively, $M(\theta)$ is the Mueller matrix that describes the scattering process[14−16], and $R(\phi)$ is the standard $4 \times 4$ matrix that rotates the reference plane, which is given by[14,15]

$$R(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\phi) & \sin(2\phi) & 0 \\ 0 & -\sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

When the $x$-$z$ plane is chosen to be the reference plane for the backscattered light, the Stokes vectors of the backscattered light in the scattering plane have to be rotated back to $x$-$z$ plane. To realize this movement of rotating back, in Ref. [14], following standard rotation about the reference plane (in the sense of direction of propagation), the Stokes vectors of the backscattered light in Eq. (1) were written as

$$S_2^{\mathrm{bs}}(\rho, \phi) = R(-\phi) S_2^{\mathrm{bs}'}(\rho, \phi), \quad (3)$$

then the effective Mueller matrix, when the $x$-$z$ plane is also chosen to be the reference plane for the backscattered light, takes the form of

$$M(\rho, \phi) = R(-\phi) L_2(\rho) R(-\phi). \quad (4)$$

It is easy to prove that the symmetries of the off-diagonal elements in Eq. (4) are not same[14].

In the following, this movement of rotating back is reconsidered. Rotation matrix $R(\phi)$ (Eq. (2)) actually connects the two Stokes vectors that describe the same polarization state of the light but with the two reference planes such that the first reference plane coincides with the second one after a rotation by an angle $\phi$. Then the inverse transformation, that gives the Stokes vector in the second reference plane in terms of its value in the first reference plane, can be obtained by the inverse operation $R^{-1}(\phi)$ ($= R(-\phi)$). Because the projections of any electric-field vector in these two defined reference plane systems are interrelated, independent of the direction of light propagation, the Stokes vectors in these two reference planes are related to each other through corresponding rotation matrix $R(\phi)$ or $R(-\phi)$. By this consideration of rotation about the reference plane, when the $x$-$z$ plane is also chosen to be the reference plane for the backscattered light, the Stokes vectors of backscattered light in Eq. (1) should now be written as

$$S_2^{\mathrm{bs}}(\rho, \phi) = R(\phi) S_2^{\mathrm{bs}'}(\rho, \phi). \quad (5)$$

The rotation angle is $\phi$, as this is the inverse transformation compared with the transformation for the incident beam. Then the effective Mueller matrix takes the form

$$M(\rho, \phi) = R(\phi) L_2(\rho) R(-\phi). \quad (6)$$

Assuming that the light is scattered in the turbid medium by spheres, the single-scattering Mueller matrix $M(\theta_i)$ in Eq. (1) takes a relatively simple form[15,16]:

$$M(\theta_i) = \begin{pmatrix} a(\theta_i) & b(\theta_i) & 0 & 0 \\ b(\theta_i) & a(\theta_i) & 0 & 0 \\ 0 & 0 & d(\theta_i) & -e(\theta_i) \\ 0 & 0 & e(\theta_i) & d(\theta_i) \end{pmatrix}, \quad (7)$$

where the four independent elements, $a$, $b$, $d$, and $e$, expressed in terms of a series of Bessell functions, are dependent on the scattering angle $\theta_i$ , the refractive indices $n_{\mathrm{med}}$ and $n_0$ of the medium and scatterers, respectively. When the particles (spheres) are sufficiently small relative to the light wavelength (in the case of Rayleigh scattering), the coefficients reduce to[12−14]

$$a(\theta) = \frac{3}{16}(1 + \cos^2\theta), \qquad b(\theta) = \frac{3}{16}(-1 + \cos^2\theta),$$

$$d(\theta) = \frac{3}{8}\cos\theta, \qquad e(\theta) = 0. \quad (8)$$

In this case it will be easy to prove that all the off-diagonal elements in Eq. (6) are symmetric (except the elements in the last row ($m_{14}$, $m_{24}$, $m_{34}$) and last column ($m_{41}$, $m_{42}$, $m_{43}$) which are all zero) (see Fig. 1(a)). Clearly, the symmetry differences are due to the different manners in dealing with the rotations with regard to the reference plane.

Based on the method of Bartel and Hielscher[10], the Monte Carlo programs for simulation of polarized light in an isotropic turbid medium are modified according to that of Wang *et al.*[17] in order to take into account of the principle of Stokes-Mueller formalism including polarization effect.

Supposing a photon is injected orthogonally into an isotropic turbid medium at the origin, which corresponds to a collimated arbitrarily narrow beam of photons. The position and direction of the photon is initialized in the global system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ with the Cartesian coordinates $(x, y, z) = (0, 0, 0)$ and the directional cosines $(ux, uy, uz) = (0, 0, 1)$. Once a step size $s_i$ is specified, the photon is ready to be moved in the medium. To keep

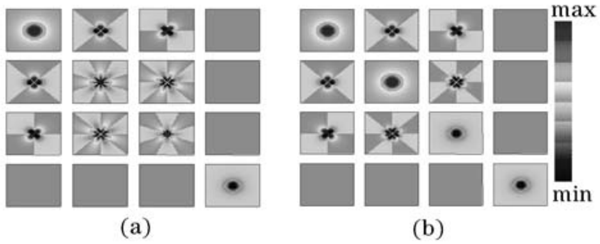


Fig. 1. Diffusely backscattered Mueller matrix elements of an isotropic medium with smaller spheres (in the case of Rayleigh scattering) based on an analytic double-scattering model. All the images displayed are $1\times1$ (mm). (a) $M(\rho,\phi)=R(\phi)L_2(\rho)R(-\phi)$; (b) $M(\rho,\phi)=R(-\phi)L_2(\rho)R(-\phi)$.

track of a polarization state as a photon undergoes multiple scattering events, it is assigned the four-component Stokes vector $S'_i$ and a local coordinate system $(\mathbf{e}_r^i,\mathbf{e}_l^i,\mathbf{e}_3^i)$ in which $S'_i$ is defined. The prime corresponds to a different reference plane from the $x$-$z$ reference plane (in the globe system $(\mathbf{i},\mathbf{j},\mathbf{k})$).

After a large number, $n$, of scattering events, Stokes vector of the exiting photons at the detector $S'_n$ is given in a randomly oriented coordinate system $(\mathbf{e}_r^n,\mathbf{e}_l^n,\mathbf{e}_3^n)$, which must be determined in terms of the global system $(\mathbf{i},\mathbf{j},\mathbf{k})$. Following the above consideration on rotating reference plane and by keeping track of the local coordinate system, the polarization state described by $S'_n$ can also be described by

$$S_n=R(\phi_1)R(\phi_2)\cdots R(\phi_{n-1})S'_n=R\left(\sum_{i=1}^{n-1}\phi_i\right)S'_n, \quad (9)$$

which corresponds to the backscattered light with respect to the $x$-$z$ reference plane in the global system $(\mathbf{i},\mathbf{j},\mathbf{k})$. Because the Stokes vectors are additive, after tracing multiple photon packets, they may simply be summed up in the two-dimensional gird system to yield the average answer of the medium. The exiting photons with any exiting angles are collected onto the detector.

The patterns of 16 matrix elements for backscattered light calculated by the double-scattering model Eq. (6), and the Eq. (4) in Ref. [14] are shown in Figs. 1(a) and (b) for the isotropic turbid media with small scattering spheres (in the case of Rayleigh scattering), respectively, where each map shows the spatial distribution of a Mueller matrix element in a designed colour scale. The parameters chosen as Ref. [12,13] for these calculations were: the absorption coefficient $\mu_\mathrm{a}=0.1$ cm$^{-1}$, the scattering coefficient $\mu_\mathrm{s}=100$ cm$^{-1}$, the refractive index of the medium $n_\mathrm{med}=1.330$, and the wavelength of the light $\lambda=594$ nm. The main differences between Figs. 1(a) and (b) are the symmetrical relationships between the off-diagonal elements and the shapes of diagonal elements $m_{22}$ and $m_{33}$. With the same parameters for the turbid medium used for the double scattering model and the thickness of the medium slab $d=0.2$ cm as Ref. [12,13], Fig. 2 illustrates the Mueller matrix elements resulting from the Monte Carlo simulations for the isotropic turbid media with small scattering spheres whose size is 204 nm in diameter (in the case of Rayleigh scattering), with Fig. 2(a) using $\phi_i$ and Fig. 2(b) using $-\phi_i$, respectively, in Eq. (9). It results in a same symmetrical

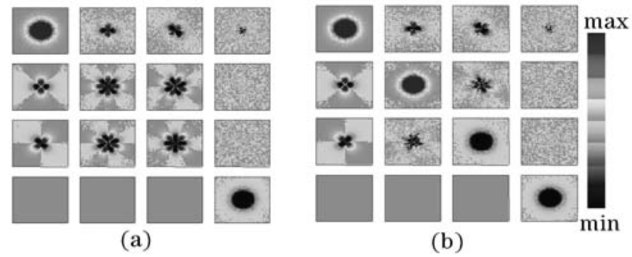


Fig. 2. Diffusely backscattered Mueller matrix elements of isotropic media with smaller spheres (in the case of Rayleigh scattering) based on Monte Carlo algorithm. All the images displayed are $1\times1$ (mm). (a)$S_n=R\left(\sum_{i=1}^{n-1}\phi_i\right)S'_n$; (b) $S_n=R\left(-\sum_{i=1}^{n-1}\phi_i\right)S'_n$.

relationship between the matrix elements in Fig. 2 to that that of double scattering model in Fig. 1. By comparing with the results of Hielscher *et al.*[6,10] for the case of Rayleigh scattering, Figs. 1(a) and 2(a) demonstrate excellent agreement with the experiment in shapes and symmetries of the images.

Actually, the symmetrical relations between the off-diagonal elements can be proved in a more general way. Let the plane of reference be (instead of the fixed $x$-$z$ plane) the plane containing the incoming and backscattered light beam. Then the Stokes vector of the backscattered light in this plane is written as

$$S^{\mathrm{bs}'}(\rho,\phi)=M'(\rho,\phi)S'_0. \quad (10)$$

By simultaneous equal rotations of the input and output reference planes (which are the same at the moment) with the same angle $\phi$ using present consideration of the reverse rotation of the reference plane, the effective Mueller matrix after the rotation will be[18]

$$M(\rho,\phi)=R(\phi)M'(\rho,\phi)R(-\phi). \quad (11)$$

Now, axial symmetry of the system is expressed by the fact that the matrix $M'(\rho,\phi)$ does not depend on $\phi$, i.e.[19],

$$M'(\rho,\phi)=L(\rho)=\sum_{n=2}^{\infty}\left(\frac{\mu_\mathrm{s}}{\mu_\mathrm{t}}\right)^{n-2}L_n(\mu_\mathrm{t}\rho), \quad (12)$$

where the term $L_n$ corresponds to the backscattered light that has been scattered $n$ times. It follows that the effective Mueller matrix takes the form

$$M(\rho,\phi)=R(\phi)L(\rho)R(-\phi). \quad (13)$$

Following the procedure in Ref. [8], it will be easy to obtain the following symmetrical relations for the off-diagonal Mueller elements in an optically inactive medium:

$$m_{12}=m_{21},\quad m_{13}=m_{31},\quad m_{14}=-m_{41},$$
$$m_{23}=m_{32},\quad m_{24}=-m_{42},\quad m_{34}=-m_{43}. \quad (14)$$

Except the off-diagonal elements in the last row ($m_{14}$, $m_{24}$, $m_{34}$) and last column ($m_{41}$, $m_{42}$, $m_{43}$) that are all anti-symmetrical, the other off-diagonal elements are

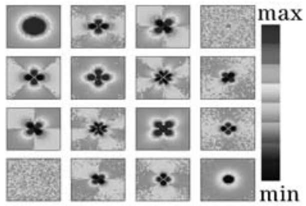


Fig. 3. Simulated backscattered Mueller matrix for the isotropic turbid medium in the case of Mie scattering based on Monte Carlo algorithm. All the images displayed are $1 \times 1$ (mm). $S_n = R\left(\sum_{i=1}^{n-1} \phi_i\right) S'_n$.

symmetrical. In fact, Eq. (6), i.e., the double scattering treatment, is a special case for Eq. (13).

Figure 3 displays the Monte Carlo simulated Mueller matrix patterns of light backscattered from an isotropic turbid medium in which the size of the scattering spheres is 700 nm in diameter (in the case of Mie scattering). Other parameters are the same as those used for Fig. 2. Comparing the results from Monte Carlo simulation, i.e. Fig. 3, with the analytical results (Eq. (14)), an excellent agreement between them is achieved. However, it should be noted that in Fig. 3 the off-diagonal elements in the last row ($m_{24}$, $m_{34}$) and last column ($m_{42}$, $m_{43}$) that are all anti-symmetrical do not agree with the experimental results from Hielscher *et al.*[6] for larger particles (in the case of Mie scattering). The reason may be that the values for these elements are approaching to zero, and as such it will not be obvious to show the correct azimuthal variations in the form of relative intensity in the experiments.

In the theoretical studies[8,14], the choice of the reference coordinate systems in polarizer may be different from the experiments[6]. The symmetry of off-diagonal elements is identical when they are associated with the reference systems. Actually, if we alternate the reference coordinate system for the backscattered light in Ref. [14] to the reference coordinate system in the experiment, which can be realized by multiplying the Mueller matrix of a mirror in Eq. (3) as well as Eq. (4), the symmetry of off-diagonal elements is same for Eqs. (4) and (6).

In conclusion, the propagation of the polarized light in the isotropic turbid media was restudied based on the analytic double-scattering model[14] and Mote Carlo algorithm[10]. The symmetrical relations of the polarization patterns of backscattered light in the isotropic media were analyzed. Results from turbid medium with smaller particles (in the case of Rayleigh scattering) and larger particles (in the case of Mie scattering) were compared with the published experimental results and discussed. The theoretical treatment presented in the current study may contribute to clarification of the rotations about reference plane in the essential physical processes of polarized light propagation in the turbid media. These rotations that should be carefully considered may result in different symmetrical relations between the off-diagonal backscattered Mueller matrix elements in theory, which has been presented in this paper.

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