Nonlinear dynamics study of an open resonant Λ-type system

Ni Cui (崔 妮), Zhengping Hong (洪正平), Hua Li (李 华), Aiyun Li (李爱云), and Xijun Fan (樊锡君)

Department of Physics, Shandong Normal University, Jinan 250014

Received March 8, 2005

A way resulting in lasing without inversion (LWI) in an open resonant Λ-type system from a nonlinear dynamics viewpoint is investigated. The destabilization of the non-lasing solution can occur not only through pitchfork bifurcation, giving rise to continuous wave LWI, but also through Hopf bifurcation, giving rise to self-pulsing LWI. This is much different from that of the corresponding closed resonant Λ-type system in which the destabilization of the non-lasing solution can occur only through pitchfork bifurcation. The effects of the unsaturated gain coefficient, cavity loss coefficient, atomic injection and exit rates on the two bifurcations are discussed.

OCIS codes: 270.1670, 270.3430.

Quantum coherence and interference in atomic systems have a number of important consequences, including lasing without inversion (LWI), the electromagnetically induced transparency, coherent population trapping, the subrecoil cooling of atoms, potentiality for sensitive measurements of magnetic fields, and so on. In particular, LWI has attracted much more attention[1−12] and have been studied in the past few decades. The open Λ-type three-level system for LWI is proposed[13,14]. From the nonlinear dynamics viewpoint, the lasing arising always corresponds to a loss of stability of the non-lasing stationary solution. Mompart *et al.*[15] have also pointed out that in the closed resonant Λ-type system destabilization of the non-lasing solution can occur only through a pitchfork bifurcation. In this paper the way resulting in the LWI in an open resonant Λ-type system from a nonlinear dynamics viewpoint is investigated. We find that the destabilization of the non-lasing solution can be obtained through both the pitchfork and Hopf bifurcations. In addition, the effects of the unsaturated gain coefficient, cavity loss coefficient, the ratio of atomic injection rates, and the atomic exit rate on the two kinds of destabilization are discussed respectively.

Figure 1 is the open resonant Λ-type system. The driving field with Rabi frequency Ω is resonant with the transition $|1\rangle \leftrightarrow |2\rangle$; the weak probe field with Rabi frequency α is resonant with the transition $|1\rangle \leftrightarrow |3\rangle$, and

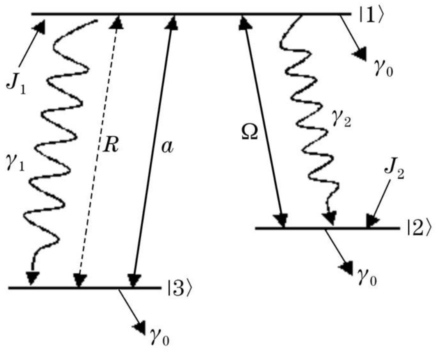

Fig. 1. An open Λ-type three-level system.

an incoherent pump field with a pumping rate R is applied between levels $|1\rangle$ and$|3\rangle$. The transition between $|2\rangle$ and$|3\rangle$ is forbidden. The resonance condition allows us to take α and Ω as real and to seek solution of the problem such that the (slowly varying) density matrix element amplitudes of the medium in the interacting picture have the form[14]

$$\rho_{12} = iy_{12}, \quad \rho_{13} = iy_{13}, \quad \rho_{23} = x_{23}, \tag{1}$$

where y_{12}, y_{13}, and x_{23} are real time-dependent variables. In our notation, if $y_{13} < 0$, the system exhibits gain for the probe field; if $y_{13} > 0$, the probe field is attenuated. If $\rho_{11} - \rho_{33} < 0$ and $y_{13} < 0$, the LWI can be possible.

In the rotating-wave, slowing varying envelope and mean-field approximations, the system is governed by the set of Maxwell-Bloch equations[14]

$$\begin{aligned}
\mathrm{d}\rho_{11}/\mathrm{d}t &= -(\gamma_1 + \gamma_2 + \gamma_0 + R)\rho_{11} + R\rho_{33} \\
&\quad +2\Omega y_{12} + 2\alpha y_{13} + J_1, \\
\mathrm{d}\rho_{22}/\mathrm{d}t &= \gamma_2\rho_{11} - \gamma_0\rho_{22} - 2\Omega y_{12} + J_2, \\
\mathrm{d}\rho_{33}/\mathrm{d}t &= (\gamma_1 + R)\rho_{11} - (\gamma_0 + R)\rho_{33} - 2\alpha y_{13}, \\
\mathrm{d}y_{12}/\mathrm{d}t &= -\Gamma_{12}y_{12} - \Omega(\rho_{11} - \rho_{22}) + \alpha x_{23}, \\
\mathrm{d}y_{13}/\mathrm{d}t &= -\Gamma_{13}y_{13} - \alpha(\rho_{11} - \rho_{33}) + \Omega x_{23}, \\
\mathrm{d}x_{23}/\mathrm{d}t &= -\Gamma_{23}x_{23} - \alpha y_{12} - \Omega y_{13}, \\
\mathrm{d}\alpha/\mathrm{d}t &= -\kappa\alpha - g y_{13},
\end{aligned} \tag{2}$$

where g is the unsaturated gain of the lasing transition (unsaturated gain coefficient), and k is the damping rate of the lasing field due to cavity loss (cavity loss coefficient). γ_1 (γ_2) denotes the decay rate from level $|1\rangle$ to $|3\rangle$ ($|2\rangle$). J_1 (J_2) is the atomic injection rate for level $|1\rangle$ ($|2\rangle$), and γ_0 is the atomic exit rate from the cavity after acting with the fields. In the following discussion, we always make $J_1 + J_2 = \gamma_0$ to keep the total number of atoms constant. In the radiative limit, the atomic polarization damping rates Γ_{12}, Γ_{13}, and Γ_{23} can be expressed as

$$\Gamma_{12} = (\gamma_1 + \gamma_2 + R)/2,$$
$$\Gamma_{13} = (\gamma_1 + \gamma_2 + 2R)/2, \quad \Gamma_{23} = R/2. \tag{3}$$

Taking $\alpha = 0$ and all the time-derivations in Eq. (2) equal to zero, the non-lasing solution is

$$y_{13} = x_{23} = 0, \quad y_{12} = \frac{\Omega}{\Gamma_{12}}(1 - \rho_{33} - 2\rho_{11}),$$
$$\rho_{11} = \frac{2(\gamma_0 + R)\Omega^2 + (R + \gamma_0)\Gamma_{12}J_1}{D},$$
$$\rho_{33} = \frac{(\gamma_1 + R)\rho_{11}}{\gamma_0 + R}, \qquad \rho_{22} = 1 - \rho_{11} - \rho_{33}, \tag{4}$$

where

$$D = [(\gamma_2 + \gamma_0)(\gamma_0 + R) + (\gamma_1 + R)\gamma_0]\Gamma_{12}$$
$$+2(\gamma_1 + 2\gamma_0 + 3R)\Omega^2. \tag{5}$$

In the above calculation we used the closure relation $\rho_{22} = 1 - \rho_{11} - \rho_{33}$. In the following, the population difference is defined as

$$n_{ij} = \rho_{ii} - \rho_{jj}, \tag{6}$$

The inversionless condition implies that $n_{31} > 0$.

From the nonlinear dynamics viewpoint, a lasing solution always corresponds to a loss of stability of the non-lasing stationary solution. Equation (2) is linearized about the Eq. (4). The resulting Jacobian matrix can be split into two independent submatrixes. One of them governs the stability of the variables α, y_{13}, and x_{23}, and therefore the generation of the lasing field. Using the same method as that in Ref. [15], the characteristic polynomial of this submatrix can be obtained as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{7}$$

with the coefficients

$$A_1 = \kappa + \Gamma_{13} + \Gamma_{23},$$
$$A_2 = \kappa(\Gamma_{13} + \Gamma_{23}) + \Gamma_{13}\Gamma_{23} + \Omega^2 + gn_{31},$$
$$A_3 = \kappa(\Gamma_{13}\Gamma_{23} + \Omega^2) + g(\Gamma_{23}n_{31} - \Omega y_{12}). \tag{8}$$

We apply the Hurwitz criteria to determine the instabilities associated with the above polynomial; if A_1, A_2, $A_3 > 0$ and $H_2(= A_1A_2 - A_3) > 0$ signify negative real parts of all eigenvalues which means stability of the non-lasing solution. From Eq. (8) we know that for the inversionless condition ($n_{31} > 0$), A_1 and A_2 are always positive. Destabilization of the trivial solution occurs through a pitchfork bifurcation (static instability) if $A_3 < 0$, or alternatively, through a Hopf bifurcation (self-pulsing instability) if $H_2 < 0$. In the second case, $\sqrt{A_2}$ gives the angular pulsation frequency of the lasing field at the destabilization point. Here, H_2 has the form

$$H_2 = (\Gamma_{13} + \Gamma_{23})[\kappa(\kappa + \Gamma_{13} + \Gamma_{23}) + \Gamma_{13}\Gamma_{23} + \Omega^2]$$
$$+g[(k + \Gamma_{13})n_{31} + \Omega y_{12}]. \tag{9}$$

Mompart *et al.*[15] pointed out that for the closed resonant Λ-type system, destabilization of the non-lasing solution can occur only through a pitchfork bifurcation giving rise to continuous wave LWI. However, we will show that, for an open Λ-type system with the appropriate value of the system parameters, destabilization of the non-lasing solution through both the pitchfork and Hopf bifurcations is possible.

The condition for pitchfork bifurcation is $A_3 < 0$. From Eq. (8) we know that, as $n_{31} > 0$ (the inversionless condition), the only term that can contribute to destabilization of the non-lasing solution is Ωy_{12}, because $y_{12} = \Omega n_{21}/\Gamma_{12} > 0$ when n_{21} is positive, i.e., there is no population inversion between levels $|1\rangle$ and $|2\rangle$. $A_3 < 0$ can be realized by adjusting properly the values of the system parameters. From Eq. (8) we know that for $\Omega = 0$ as well as for very large values of Ω^2, A_3 is positive and the non-lasing solution is stable. So the value of the driving field intensity is limited in some regions for obtaining $A_3 < 0$. Figures 2(a)—(d) illustrate different curves for $A_3 = 0$ as a function of the parameters Ω and R for different values of the unsaturated gain coefficient g, the cavity loss coefficient k, the ratio of the atomic injection rates $S(= J_1/J_2)$ and the atomic exit rate γ_0, respectively. The values of g are measured in MHz^2 , and other parameters are measured in MHz. For a given value of a parameter, for example, g in Fig. 2(a), LWI is obtained within a closed curve in the Ω-R plane ($A_3 < 0$). Outside this curve the non-lasing solution is stable. Figure 2 shows that destabilization of the non-lasing solution requires a good cavity (k is smaller) and a higher g; a smaller value of S and a larger value of γ_0 are favorable. When $\gamma_0 = S = 0$, i.e., the open system changes into a closed system, the region getting LWI in the Ω-R plane is still present, i.e., the pitchfork bifurcation occurs in a closed Λ-type system and this consists with the result given by Mompart *et al.*[15] for the closed Λ-type system.

From Eq. (9) we know that, as $n_{31} > 0$ (the inversionless condition) , in order to satisfy the condition for Hopf bifurcation ($H_2 < 0$), $y_{12} < 0$ is necessary, and this

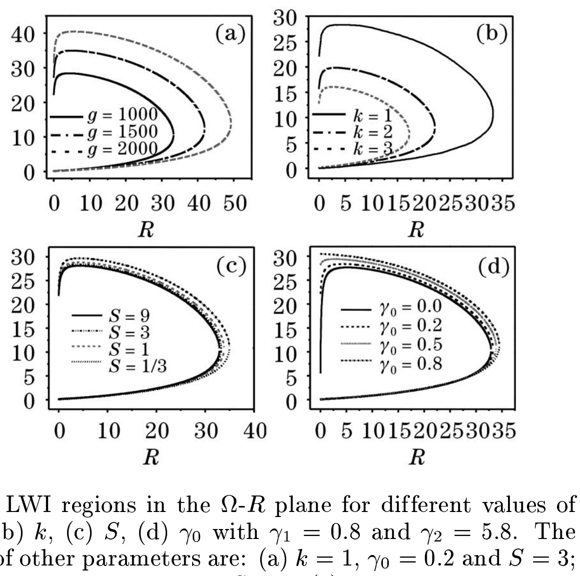

Fig. 2. LWI regions in the Ω-R plane for different values of (a) g, (b) k, (c) S, (d) γ_0 with $\gamma_1 = 0.8$ and $\gamma_2 = 5.8$. The values of other parameters are: (a) $k = 1$, $\gamma_0 = 0.2$ and $S = 3$; (b) $g = 1000$, $\gamma_0 = 0.2$, and $S = 3$; (c) $g = 1000$, $k = 1$, and $\gamma_0 = 0.2$; (d) $g = 1000$, $k = 1$, and $S = 3$.

means the population inversion between levels $|1\rangle$ and $|2\rangle$ arising (see Eq. (4)). We can suitably select values of the system parameters including S and γ_0 to make $y_{12} < 0$, thereby $H_2 < 0$ in the open Λ-type system. In the corresponding closed Λ-type system, however, the population inversion between levels $|1\rangle$ and $|2\rangle$ is not possible, so that Hopf bifurcation does not arise.

Similar to the discussion about pitchfork bifurcation, we analyze the effects of g, k, S, and γ_0 on destabilization of the non-lasing solution by the numerical calculation results. Figures 3(a)—(d) illustrate different curves for $H_2 = 0$ as a function of the parameters Ω and R for different values of g, k, S, and γ_0, respectively. Same as Fig. 2, here the values of g are measured in MHz^2, and the other parameters are measured in MHz. For a given value of a parameter, for example, g in Fig. 3(a), LWI is obtained within a closed curve in the Ω-R plane ($H_2 < 0$). Outside this curve the non-lasing solution is stable.

Comparing Fig. 2 with Fig. 3, we find that: 1) The effects of g and k on the region with LWI in the Ω-R plane are the same for the two types of the bifurcations, i.e., with $g(k)$ increasing, the region with LWI becomes larger (smaller). 2) Different from the situation in the pitchfork bifurcation, in the Hopf bifurcation the region with LWI decreases with both S and γ_0 decreasing, and when $S = \gamma_0 = 0$ the region with LWI will disappear. This conclusion is easy to understand from the physical viewpoint. When $S = \gamma_0 = 0$, that is when the open scheme changes into a closed scheme, the Hopf bifurcation should be absent. 3) To obtain LWI, in the Hopf bifurcation case, the value range of Ω is much larger than that of R; however, in the pitchfork bifurcation case the value range of Ω is approximately equal to that of R.

In conclusion, in this paper we investigated the way resulting in LWI in an open resonant Λ-type system from a nonlinear dynamics viewpoint. We find that destabilization of the non-lasing solution can occur not only through a pitchfork bifurcation giving rise to continuous-wave LWI but also through a Hopf bifurcation giving rise to self-pulsing LWI. This conclusion is much different from that obtained in a corresponding closed resonant Λ-type system. The two bifurcations are mutually exclusive since they are primarily determined by the opposite population situation, that is inversion or inversionless. The Hopf bifurcation can occur in the open Λ-type system but cannot occur in the corresponding closed Λ-type system, the essential cause is that in the open system we can change the population situation (inversion or inversionless) between the two levels coupled by the driving field through varying S and γ_0, but in the closed system this is impossible. The numerical calculation results show that: 1) For both the pitchfork and Hopf bifurcations, a higher g and a smaller k are favorable to obtain destabilization of the non-lasing solution. 2) The effects of S and γ_0 are related with the type of the bifurcations. In the pitchfork bifurcation case, a larger value of γ_0 and a smaller value of S are favorable to obtain the destabilization; when $\gamma_0 = S = 0$, the region getting LWI in the Ω-R plane is still present, i.e., the pitchfork bifurcation occurs in a closed Λ-type system and this consists with the result given Mompart *et al.*[15]. In the Hopf bifurcation case, the larger values of γ_0 and S are favorable for obtaining the destabilization; when $\gamma_0 = S = 0$, the region getting LWI in the Ω-R plane will disappear, i.e., the Hopf bifurcation does not exist in a closed Λ-type system and this also consists with the result given by Mompart *et al.*[15].

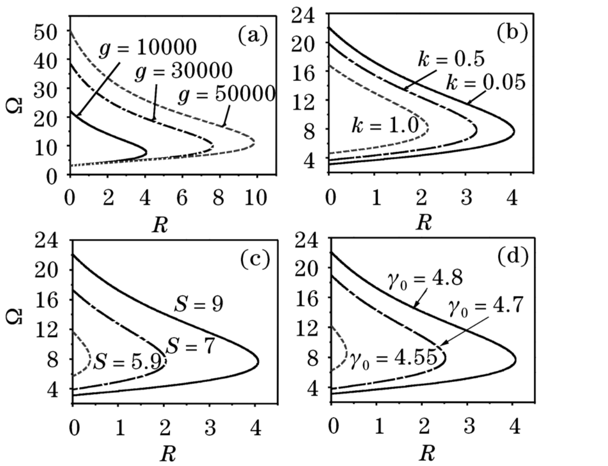

Fig. 3. LWI regions in the Ω-R plane for different values of (a) g, (b) k, (c) S, (d) γ_0 with $\gamma_1 = 5.8$ and $\gamma_2 = 0.8$. The values of other parameters are as follows: (a) $k = 0.05$, $\gamma_0 = 4.8$ and $S = 9$; (b) $g = 10000$, and $\gamma_0 = 4.8$ and $S = 9$; (c) $g = 10000$, $k = 0.05$ and $\gamma_0 = 4.8$; (d) $g = 10000$, $k = 0.05$ and $S = 9$.

This work was supported by the Key Laboratory for High Intensity Optics, Shanghai Institute of Optics and Fine Mechanics, Chinese Academy of Sciences. X. Fan is the author to whom the correspondence should be addressed, his e-mail address is fanxj108@beelink.com.

References

1. J. Mompart and R. Corbalán, J. Opt. B: Quantum Semiclass. Opt. **2,** R7 (2000).
2. Y. Bai, H. Guo, H. Sun, D. Han, C. Liu, and X. Chen, Phys. Rev. A **69,** 043814 (2004).
3. Y. Golubev, O. Kocharovskaya, Y. Rostovtsev, and M. O. Scully, J. Opt. B: Quantum Semiclass. Opt. **6,** 309 (2004).
4. X. Fan, C. Liu, S. Tian, J. Li, M. Zhu, N. Cui, and S. Gong, J. Mod. Opt. **51,** 399 (2004).
5. X. Fan, H. Xu, S. Tian, D. Tong, and Z. Xu, Opt. Commun. **241,** 399 (2004).
6. X. Fan, H. Xu, S. Tian, and D. Tong, Chin. Opt. Lett. **2,** 56 (2004).
7. X. Fan, M. Zhu, Z. Hong, S. Gong, and Z. Xu, Chin. Opt. Lett. **2,** 673 (2004).
8. D. Braunstein and R. Shuker, Phys. Rev. A **68,** 013812 (2003).
9. C. Liu, S. Gong, Z. Zheng, and Z. Xu, Chin. Opt. Lett. **1,** 423 (2003).
10. X. Fan, C. Liu, S. Tian, H. Xu, M. Zhu, and J. Li, J. Mod. Opt. **50,** 1763 (2003).
11. O. Kocharovskaya, A. B. Matsko, and Y. Rostovtsev, Phys. Rev. A **65,** 013803 (2001).
12. X. Hu, J. Xiong, and J. S. Peng, Eur. Phys. J. D **13,** 401 (2001).
13. S.-Y. Zhu, D.-Z. Wang, and J.-Y. Gao, Phys. Rev. A **55,** 1339 (1997).
14. X. Fan, S. Tian, J. Li, J. Liu, and C. Bai, Acta Phys. Sin. (in Chinese) **49,** 1719 (2000).
15. J. Mompart, C. Peters, and R. Corbalán, Phys. Rev. A **57,** 2163 (1998).