

Laser controlling chaotic region of a two-component Bose-Einstein condensate

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Received January 5, 2005

For a weakly and periodically driven two-component Bose-Einstein condensate (BEC) the Melnikov chaotic solution and boundedness conditions are derived from a direct perturbation theory that leads to the chaotic regions in the parameter space. Differing from the usual results, the chaotic regions depend on the initial conditions directly and can be controlled by setting the initial phase and modulating the frequency and amplitude of laser. It is demonstrated that the order-chaos transitions can be controlled by tuning the laser frequency only.

OCIS codes: 140.1340, 020.0020.

Because of the experimental realization of Bose-Einstein condensation in weakly interacting atomic systems, many fundamental works for Bose-Einstein condensates (BECs) have been carried out. One of the most interesting future prospects for the BECs is their application as a source of coherent matter waves^[1-4] in atomic optics and atomic interferometry. Meanwhile, the analogies between coherent matter waves and coherent photons have been discussed as an “atom laser”^[5-9]. Particularly, Bloch *et al.* reported on the atom optical manipulation of an atom laser beam, realized a versatile atom optical element, and demonstrated its reflection, split, and focus^[10]. On the other hand, laser is comprehensively applied in atom-molecule system, a lot of works about laser-atom interactions have been reported^[11-14]. Interactions between the BECs and standing wave laser fields have also been investigated^[15-20].

Recently, much interest was focused on the chaotic behavior of two coupled BECs with periodical or kicked trapping potentials^[21-28]. In this paper, we investigate the chaotic regions of a BEC system with two hyperfine states in a single well. According to the definition of Melnikov chaos and using the direct perturbation technique^[29-32], we analytically solve the system by treating the time-dependent potential and population transfer as perturbations, obtaining the population distribution of BEC, which is analytically bounded but numerically unbounded. From the analytical boundedness condition, we get the superior and inferior limits of the first correction, and discuss the chaotic regime in parameter space through the Melnikov’s chaos criterion. The results show a new and interesting property, that is, the chaotic regions directly depend on the initial conditions. The region widths can be adjusted by modulating the amplitude of laser and the order-chaos transitions can be exactly controlled by tuning the laser frequency.

We consider the two hyperfine states $|1\rangle = |F = 1, m = -1\rangle$, $|2\rangle = |F = 2, m = 1\rangle$ of a BEC in a single well^[33], where F and m are the total angular momentum and magnetic quantum numbers^[10], respectively. Under the condition of a weak external magnetic field, the two hyperfine states are coupled by a two-photon pulse. In the rotating frame, ignoring the high-frequency terms in

the atom-field interaction, the coupled two-component BEC system is governed by a pair of coupled Gross-Pitaevskii equations (GPEs)^[33-35]

$$\begin{aligned} i\frac{\partial\Psi_1(\vec{r},t)}{\partial t} &= \left[H_1^0 + H_1^{MF} + \frac{d}{2} \right] \Psi_1(\vec{r},t) + \Omega\Psi_2(\vec{r},t), \\ i\frac{\partial\Psi_2(\vec{r},t)}{\partial t} &= \left[H_2^0 + H_2^{MF} - \frac{d}{2} \right] \Psi_2(\vec{r},t) + \Omega\Psi_1(\vec{r},t), \end{aligned} \quad (1)$$

where the “nature” unit have been adopted. The frequency of magnetic trap is ω , so the time is in the units of $1/\omega$, and the energy is in the units of $\hbar\omega$. Here $\Psi_1(\vec{r},t)$ and $\Psi_2(\vec{r},t)$ are the wave functions of states $|1\rangle$ and $|2\rangle$, respectively. They are normalized to give the populations $N_i(t)$ for the state i , where the total atomic number $N = N_1 + N_2$ is a constant. The free evolution Hamiltonians are $H_i^0 = \frac{p_i^2}{2} + V_i(r)$ ($i = 1, 2$), the mean-field interaction Hamiltonians read $H_i^{MF} = \lambda_{ii}|\Psi_i|^2 + \lambda_{ij}|\Psi_j|^2$ ($i, j = 1, 2$), and the atom-atom interaction strength is expressed by $\lambda_{ij} = a_{ij}/z_{\text{sho}}$, which is proportional to the scattering length a_{ij} and inverse proportional to the harmonic oscillation length $z_{\text{sho}} = \sqrt{\hbar/(m\omega)}$ with m being the atomic mass. By Ω we mean the Rabi frequency and $d = \omega' - \omega_0$ the detuning with ω' being the driven frequency of the two-photon pulse and ω_0 the inter-electronic transition frequency. We set the wave functions $\Psi_i(\vec{r},t) = \Phi_i(t)\psi_i(\vec{r})$, $\Phi_i(t) = \sqrt{N_i(t)}e^{i\alpha_i(t)}$ ($i = 1, 2$) is the time dependence in the macroscopic quantum wave functions, and $\psi_i(\vec{r})$ can be interpreted as the spatial distribution of the i -th component. The symbols $N_i(t)$ and $\alpha_i(t)$ represent the population and phase of the i -th component, respectively. Applying such wave function to Eq. (1) and integrating over spatial coordinates reveal that $\Phi_i(t)$ for $i = 1, 2$ is described by the nonlinear equations^[34,35]

$$i\frac{\partial\Phi_1(t)}{\partial t} = \left[E_1^0 + \frac{d}{2} + N_1U_{11} + N_2U_{12} \right] \Phi_1(t) + K\Phi_2(t),$$

$$i \frac{\partial \Phi_2(t)}{\partial t} = \left[E_2^0 - \frac{d}{2} + N_2 U_{22} + N_1 U_{12} \right] \Phi_2(t) + K \Phi_1(t), \tag{2}$$

with $E_i^0 = \int \psi_i^*(\vec{r}) H_i^0 \psi_i(\vec{r}) d\vec{r}$ representing the zero point energy, $U_{ij} = \frac{a_{ij}}{z_{\text{sho}}} \int |\psi_i(\vec{r})|^2 |\psi_j(\vec{r})|^2 d\vec{r} = U_{ji}$ describing the mean-field interaction between atoms, and $K = \Omega \int \psi_2^*(\vec{r}) \psi_1(\vec{r}) d\vec{r} = \Omega \int \psi_1^*(\vec{r}) \psi_2(\vec{r}) d\vec{r}$ being the population transfer between two BEC states. Introducing the relative population

$$\eta(t) \equiv [N_2(t) - N_1(t)]/N, \tag{3}$$

and relative phase

$$\phi(t) \equiv \alpha_2(t) - \alpha_1(t), \tag{4}$$

and substituting the above expressions into Eq. (2) yields^[33]

$$\dot{\eta} = -2K \sqrt{1 - \eta^2} \sin \phi, \tag{5}$$

$$\dot{\phi} = \gamma + U\eta + 2K\eta(1 - \eta^2)^{-\frac{1}{2}} \cos \phi, \tag{6}$$

where $\gamma = E_1^0 - E_2^0 + N \frac{U_{11} - U_{22}}{2} + d$, $U = -N(U_{22} + U_{11} - 2U_{12})/2$. $\dot{\eta}$ denotes the time derivative of the relative population and $\dot{\phi}$ represents the time derivative of the relative phase. Equations (5) and (6) imply that η and ϕ are the formally canonical momentum and its conjugate coordinate, which satisfy the formally canonical equations $\dot{\eta} = -\frac{\partial H}{\partial \phi}$, $\dot{\phi} = \frac{\partial H}{\partial \eta}$ for the conserved Hamiltonian

$$H = \gamma\eta + \frac{U}{2}\eta^2 - 2K \sqrt{1 - \eta^2} \cos \phi. \tag{7}$$

From it, we can see that the dynamic evolution of this system are closely related to the population transfer K , relative energy γ , and mean-field parameter U . In the following section, we will discuss the chaotic regions of the system and the control to them by using a laser field.

The fundamental property of chaos is that the dynamic behaviors are unpredictable for a deterministic system, it depends on the initial conditions and parameters of the system sensitively. Generally speaking, chaos appears only for a certain realm of parameters. So determining the chaotic region is very impotent. Much work have been done with harmonic potential and periodical perturbation^[21-28,36,37]. In the following, we investigate the chaos regime of the system with periodical potential and population transfer. In experiment, we can adjust the laser position and intensity to make the time-dependent potential as $V_{\text{trap}}(\vec{r}, t) = V_0(\vec{r}) + V_1(\vec{r}) \sin(\omega_1 t + \varphi)$ ^[36,37]. Substituting it into the expressions of parameters E_i^0 and γ , we get the time-dependent relative energy as $\gamma(t) = \gamma_0 + \gamma_1 \sin(\omega_1 t + \varphi)$ with φ being the initial phase and ω_1 the oscillation frequency of laser position. On the other hand, in a low-frequency laser field, the Rabi frequency can be set in the form^[38] $\Omega(t) = \sqrt{W_{10}^2 + 4|\Lambda|^2 \cos^2 \omega_2 t}$, where $W_{10} = W_1 - W_0$ is the energy difference with W_1 and W_0 being the energies of states $|2\rangle$ and $|1\rangle$ respectively, Λ denotes the

coupling energy. When the coupling energy is very small in comparison with the energy difference W_{10} , substituting the time-dependent Rabi frequency into the expression of K , the population transfer approximately reads as $K(t) = K_0 + K_1 \cos(2\omega_2 t)$. Applying the $\gamma(t)$ and $K(t)$ to Eqs. (5) and (6), the later becomes the time-dependent system one. We know, when parameters $K(t)$ and $\gamma(t)$ are periodical functions of time, it is quite difficult to analytically solve this group of equations. However, for a weak V_1 and the small differences $E_1^0 - E_2^0$ and $U_{11} - U_{22}$ we have $|\gamma_{0,1}| \sim |K_1/K_0| \ll 1$ and can seek a perturbed solution. It is well-known that Melnikov's chaos is just the chaos of perturbed solution^[21]. To do this, we expand η , ϕ , H to the first order

$$\eta = \eta_0 + \eta_1, \quad |\eta_0| \gg |\eta_1| \sim |\gamma_{0,1}| \sim |K_1|,$$

$$\phi = \phi_0 + \phi_1, \quad |\phi_0| \gg |\phi_1| \sim |\gamma_{0,1}| \sim |K_1|,$$

$$H = H_0 + H_1, \quad |H_0| \gg |H_1| \sim |\gamma_{0,1}| \sim |K_1|, \tag{8}$$

where η_0, ϕ_0, H_0 denote the unperturbed quantities and η_1, ϕ_1, H_1 are the first-order corrections. Inserting the above expressions into Eqs. (5) and (6) yields the zero-order equation

$$\ddot{\eta}_0 = (-4K_0^2 + H_0 U) \eta_0 - \frac{U^2}{2} \eta_0^3,$$

$$H_0 = \frac{U}{2} \eta_0^2 - 2K_0 \sqrt{1 - \eta_0^2} \cos \phi_0 = \text{constant}, \tag{9}$$

and first-order equation

$$\ddot{\eta}_1 = (H_0 U - 4K_0^2) \eta_1 - \frac{3}{2} U^2 \eta_0^2 \eta_1 + \varepsilon_1(t),$$

$$\varepsilon_1(t) = \frac{-2K_1 \omega_2 \sin(2\omega_2 t)}{K_0} \eta_0 - 8K_0 K_1 \cos(2\omega_2 t) \eta_0$$

$$+ H_0 \gamma(t) + H_1 U \eta_0 - \frac{3}{2} U \eta_0^2 \gamma(t),$$

$$H_1 = \int \left[\cos(\omega_1 t + \varphi) \gamma_1 \omega_1 \eta_0 - \frac{2H_0 K_1 \omega_2}{K_0} \sin(2\omega_2 t) + \frac{U \eta_0^2 K_1 \omega_2 \sin(2\omega_2 t)}{K_0} \right] dt. \tag{10}$$

The unperturbed Eq. (9) has the well-known homoclinic solution for $(H_0 U - 4K_0^2) > 0$

$$\eta_0(t) = 2 \sqrt{(H_0 U - 4K_0^2)/U^2} \text{sech} \xi,$$

$$\xi = \sqrt{(H_0 U - 4K_0^2)} t + c,$$

$$c = \text{Arsech}[\eta_0(t_0)/2] \sqrt{(H_0 U - 4K_0^2)/U^2}$$

$$- \sqrt{(H_0 U - 4K_0^2)} t_0, \tag{11}$$

with t_0 being the initial time. Applying Eq. (11) to Eq.

(10), we construct the general solutions^[26]

$$\eta_1 = \eta_1'' \int_C \eta_1' \varepsilon_1(t) dt - \eta_1' \int_D \eta_1'' \varepsilon_1(t) dt, \quad (12)$$

where C and D are the integration constants, η_1' and η_1'' are two linearly independent solutions of Eq. (10) for $\varepsilon_1(t) = 0$

$$\begin{aligned} \eta_1' &= \eta_0 = -\frac{2(H_0U - 4K_0^2)}{U} \operatorname{sech} \xi \tanh \xi, \\ \eta_1'' &= \eta_1' \int (\eta_1')^{-2} dt \\ &= \frac{U \operatorname{sech} \xi}{4\sqrt{(UH_0 - 4K_0^2)^3}} (2 - 3\xi \tanh \xi - \sinh^2 \xi). \end{aligned} \quad (13)$$

Clearly, η_1' is a bounded function, but η_1'' increases exponentially with the increase of ξ . So, the corrected solution of Eq. (12) is bounded if and only if the condition^[26]

$$I_{\pm} = \lim_{t \rightarrow \pm\infty} \int_A^{\xi} \eta_1' \varepsilon_1(t) dt = 0, \quad (14)$$

is satisfied. Under the condition, applying Eqs. (11), (13), and l'Hôpital rule we give the superior and inferior limits of the corrected solution as $\lim_{t \rightarrow \pm\infty} \eta_1 = H_0(\gamma_0 + \gamma_1)$, $\lim_{t \rightarrow \pm\infty} \eta_1 = 0$. Here, we suppose that all of the parameters are positive. From above inequalities we know that for a long enough time one cannot obtain the exact value of population, but the superior and inferior limits of the first correction offer us the finite regime of motion. The value of population varies in the realm and never goes beyond this one. Taking $I_- - I_+ = 0$ from Eq. (14), we get the well-known Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} \eta_1' \varepsilon_1(t) dt = 0, \quad (15)$$

which indicates the existence of chaos and the chaotic region in parameter space^[21,24]. Combining Eqs. (13), (10) with Eq. (15), we obtain the parameter relation

$$\frac{\sin(\alpha c)}{\cos(\lambda c - \varphi)} = \frac{K_0 \gamma_1 \lambda}{2K_1 \omega_2 \alpha} \operatorname{sech}\left(\frac{\pi}{2}\lambda\right) \sinh\left(\frac{\pi}{2}\alpha\right), \quad (16)$$

where $\alpha = 2\omega_2/\sqrt{H_0U - 4K_0^2}$, $\lambda = \omega_1/\sqrt{H_0U - 4K_0^2}$. This leads to different parameter regions of Melnikov chaos for the two cases.

Case 1: For the initial laser phase $\varphi = 0$ and the parameter relation $\alpha/\lambda = 2^n$, e.g., $\omega_2/\omega_1 = 2^{n-1}$ for $n = 0, 1, 2, \dots$, the left side of Eq. (16) becomes $\sin(\alpha c)/\cos(\lambda c) = \sin(2^n \lambda c)/\cos(\lambda c) = 2^n A$, $A = \sin(\lambda c) \cos(2\lambda c) \cos(2^2 \lambda c) \dots \cos(2^{n-1} \lambda c)$. Equating this with the right side of Eq. (16) and noticing $|A| \leq 1$, we reach the chaotic region

$$\begin{aligned} K_1 &\geq \frac{\gamma_1 K_0}{2^{2n+1} \omega_2} \operatorname{sech}\left(\frac{\pi \omega_2}{2^n \sqrt{H_0U - 4K_0^2}}\right) \\ &\times \sinh\left(\frac{\pi \omega_2}{\sqrt{H_0U - 4K_0^2}}\right). \end{aligned} \quad (17)$$

From above expression we find that the chaotic region will be widened with the increase of n or the frequency ratio ω_2/ω_1 . Meanwhile, the extent of chaotic region is also associated with the parameters U , K_0 and relative energy γ_1 .

Case 2: For the initial laser phase $\varphi = \frac{\pi}{2}$, Eqs. (16) is

$$\frac{\sin(\alpha c)}{\sin(\lambda c)} = \frac{K_0 \gamma_1 \lambda}{2K_1 \omega_2 \alpha} \operatorname{sech}\left(\frac{\pi \lambda}{2}\right) \sinh\left(\frac{\pi \alpha}{2}\right). \quad (18)$$

In this case, the chaotic regions depend on the integration constant c , which corresponds to the initial conditions in Eq. (11). Defining $B = \sin(\alpha c)/\sin(\lambda c)$, when αc and λc are in the same quadrant, we can easily determine the chaotic regions. For a fixed c from the definitions of α and λ we can set $\alpha \approx \lambda$ by adjusting one of the frequencies to obey $\omega_1 \approx 2\omega_2$ such that αc and λc are in the same quadrant indeed. When they are in the first or third quadrant, $\alpha c, \lambda c \in (0, \frac{\pi}{2})$ or $\in (\pi, \frac{3\pi}{2})$, if $2\omega_2 \geq \omega_1$ ($\alpha \geq \lambda$), we have $|B| \geq 1$ such that Eq. (18) implies the chaotic region

$$\begin{aligned} K_1 &\leq \frac{K_0 \gamma_1 \omega_1}{4\omega_2^2} \operatorname{sech}\left(\frac{\pi \omega_1}{2\sqrt{H_0U - 4K_0^2}}\right) \\ &\times \sinh\left(\frac{\pi \omega_2}{\sqrt{H_0U - 4K_0^2}}\right). \end{aligned} \quad (19)$$

Similarly, in the second or fourth quadrant, $\alpha c, \lambda c \in (\frac{\pi}{2}, \pi)$ or $\in (\frac{3\pi}{2}, 2\pi)$, the same values of the parameters ω_2 and ω_1 make the inequality $0 < |B| \leq 1$ and the chaotic region

$$\begin{aligned} K_1 &\geq \frac{K_0 \gamma_1 \omega_1}{4\omega_2^2} \operatorname{sech}\left(\frac{\pi \omega_1}{2\sqrt{H_0U - 4K_0^2}}\right) \\ &\times \sinh\left(\frac{\pi \omega_2}{\sqrt{H_0U - 4K_0^2}}\right). \end{aligned} \quad (20)$$

From Eqs. (19) and (20) we can see that when the sign “ = ” is taken, they give the boundary of the chaotic regions in the parameter space. In the case of Eq. (19), K_1 below the boundary is associated with the chaotic region and K_1 above the boundary is corresponded to the region of regular motions (regular region). However, the chaotic region and regular one are exchanged for the case of Eq. (20). It is worth noting that the changes of chaotic regions may be completed, through two different methods. One is fixing the parameter relation $2\omega_2 \geq \omega_1$ and adjusting the initial constant c . But since the constant c is associated with the initial value of η_0 and the chaotic system depends on the initial conditions sensitively, this adjustment cannot be precisely performed. To control the chaotic regions experimentally, we should adopt another method, namely tune the controllable physical parameters ω_2 for a fixed c value. From the derivation of Eqs. (19) and (20) we find that if the system is in the chaotic region initially, the parameter adjustments from $2\omega_2 \geq \omega_1$ to $2\omega_2 \leq \omega_1$ lead to the transition from chaos

to regular motion. Therefore, we can exactly control the chaotic motion by use of this method. In the adjustment processes, the constant c should be fixed and the relation $2\omega_2 \approx \omega_1$ must be kept for holding the same quadrant of $a c$ and λc . On the other hand, the areas of the chaotic regions vary with the parameters ω_1 , ω_2 , and γ_1 etc.. We can increase or decrease the areas by adjusting the controllable parameters.

In conclusion, we have studied the chaotic region of two-component BEC system with periodically perturbed external potential and population transfer. Applying the direct perturbation method, we obtain the chaotic solution which is unpredictable but bounded. The theoretical analysis offers the superior and inferior limits of the chaotic solution so we can assert that the chaotic motion occurs in a range between the superior and inferior limits. The boundedness condition of the chaotic solution results in the chaotic regions on the parameter space. It is demonstrated that by tuning the laser frequency ω_2 and other parameters we can change the areas of the chaotic regions and control the chaos-order transitions.

This work was supported by the National Natural Science Foundation of China under Grant No. 10275023. W. Hai is the author to whom the correspondence should be addressed, his e-mail address is adcve@public.cs.hn.cn.

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