# Exact chirped multi-soliton solutions of the nonlinear Schrödinger equation with varying coefficients

**Ruiyu Hao (郝瑞宇)[1,3,4], Lu Li (李 录)[2,3,4], Rongcao Yang (杨荣草)[1,3,4], Zhonghao Li (李仲豪)[1,3,4], and Guosheng Zhou (周国生)[1,3,4]**

[1] *Department of Electronics and Information Technology*

[2] *Department of Physics, and Institute of Theoretical Physics*

[3] *State Key Laboratory of Quantum Optics and Quantum Optics Devices*

[4] *The State Key Subject of Optics, Shanxi University, Taiyuan 030006*

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In this letter, exact chirped multi-soliton solutions of the nonlinear Schrödinger (NLS) equation with varying coefficients are found. The explicit chirped one- and two-soliton solutions are generated. As an example, an exponential distributed control system is considered, and some main features of solutions are shown. The results reveal that chirped soliton can all be nonlinearly compressed cleanly and efficiently in an optical fiber with no loss or gain, with the loss, or with the gain. Furthermore, under the same initial condition, compression of optical soliton in the optical fiber with the loss is the most dramatic. Also, under nonintegrable condition and finite initial perturbations, the evolution of chirped soliton has been demonstrated by simulating numerically.

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The nonlinear Schrödinger (NLS) equation plays a vital role in the studies of modern nonlinear science. The best known solutions of the NLS equation are those for solitary waves[1−3], or solitons. Hasegawa and Tappert[4] firstly predicted optical solitons theoretically, and Mollenauer *et al.*[5], who observed them experimentally, made solitons a realistic tool for this cause. Today, optical solitons, localized-in-time optical pulses, will play an important role in the field of high data transmission and optical computing in the near future.

A concept called soliton control revolutionizes a new and exciting area in the application of solitons, and has been studied extensively because of potential values[6−8]. We would note the fact that the first soliton dispersion management experiment in a fiber with hyperbolically decreasing group velocity dispersion was realized as early as in 1991 by Dianov group at the General Physics Institute[9]. The various soliton management regimes have also been predicted by many authors[3,10,11] using different methods theoretically. In general, the problem of soliton control in the nonlinear systems is described by the NLS equation model with variable coefficients as

$$iA_z + \frac{1}{2}D_1(z)A_{TT} + R_1(z)\left|A\right|^2 A + i\Gamma_1(z)A = 0, \quad (1)$$

where $A \equiv A(z,T)$ is the complex envelope of the electrical field in a comoving frame, $A_z \equiv \frac{\partial A}{\partial z}$, $A_{TT} \equiv \frac{\partial^2 A}{\partial T^2}$, $D_1(z)$ represents the group velocity dispersion, $R_1(z)$ is the nonlinearity parameter, and $\Gamma_1(z)$ is the amplification or absorption coefficient, which are functions of the normalized propagation distance $z$, $T$ is the retarded time. Equation (1) describes the amplification or absorption of pulses propagating in a single mode optical fiber with distributed dispersion and nonlinearity. In practical applications, the model is of primary interest not only for the amplification and compression of optical solitons in inhomogeneous systems, but also for the core of managed soliton. Recently, the applications of Eq. (1) with various forms have been studied in various papers. In these literatures, exact one-soliton solution was obtained by Serkin, Belyaeva, and Hasagawa from the integrable point of view, and exact self-similar solutions have also been given by Kruglov, Peacock, and Harvey.

In this letter, we investigate the generalized NLS equation with variable coefficients from the integrable point of view, and present exact chirped one- and two-soliton solutions. The importance of the results presented here is twofold. First, exact chirped soliton solutions to the generalized NLS equation with variable coefficients are obtained in a simple way. The finding of a new mathematical algorithm to discover soliton solutions in nonlinear dispersive systems with spatial parameter variations is helpful on future research. Second, these results are useful not only in the design of transmission lines with soliton management, but also in some experiments of other problems.

Equation (1) has the solution[12,13]

$$A(z,T) = w(z,T)\exp[iC(z)T^2/2], \quad (2)$$

where $C(z)$ represents the frequency chirping. Substituting Eq. (2) into Eq. (1), one can derive the following equation

$$i(\frac{\partial w}{\partial z} + D_1CT\frac{\partial w}{\partial T}) + \frac{D_1}{2}\frac{\partial^2 w}{\partial T^2} + R_1|w|^2 w - \frac{1}{2}(\dot{C} + D_1C^2)T^2 w + i(\frac{D_1C}{2} + \Gamma_1)w = 0, \quad (3)$$

where $\dot{C}$ represents the derivative of $C$ with respect to $z$. Now we introduce a new coordinate $t$ and the amplitude function $a(z)$ such as

$$t = p(z)T = T\exp[-\int_0^z D_1(\zeta)C(\zeta)\mathrm{d}\zeta], \quad (4)$$

$$w(z,T) = a(z)u(z,t). \quad (5)$$

Thus Eq. (3) becomes

$$i\frac{\partial u}{\partial z}+\frac{D(z)}{2}\frac{\partial^2 u}{\partial t^2}+R(z)|u|^2u+M(z)t^2u+iF(z)u=0, \quad (6)$$

where

$$\begin{cases} D(z)=p^2D_1, R(z)=a^2R_1, \\ M(z)=-\frac{\dot{C}+D_1C^2}{2p^2}, \\ F(z)=\frac{\dot{a}}{a}+\frac{1}{2}D_1C+\Gamma_1, \end{cases} \quad (7)$$

where $\dot{a}$ represents the derivative of $a$ with respect to $z$. Generally, Eq. (6) is not integrable. To solve the Eq. (6), we consider the following relationship:

$$F(z)=\frac{1}{2}\frac{W[R(z),D(z)]}{R(z)D(z)} \quad , \quad (8)$$

$$\dot{C}+D_1C^2=0 \quad , \quad (9)$$

where

$$W[R(z),D(z)]=DR_z-RD_z.$$

Here, we can directly use the Eqs. (7) and (11) of Ref. [14], and obtain one- and two-soliton solutions for Eq. (6). Then, through the transformation of Eqs. (4), (5), and (2), when

$$\Gamma_1(z)=\frac{1}{2}\{\frac{D_1}{\int D_1\mathrm{d}\varsigma}+\frac{W[R_1,D_1]}{R_1D_1}\}, \quad (10)$$

where arbitrary integrable constant is zero, we can obtain exact chirped one- and two-soliton solutions of Eq. (1) in explicit forms as

$$A_1=u_1\exp[-\frac{1}{2}\int_0^z\frac{D_1}{\int D_1\mathrm{d}\varsigma}\mathrm{d}\varsigma+i\frac{T^2}{2\int D_1\mathrm{d}\varsigma}], \quad (11)$$

$$A_2=u_2\exp[-\frac{1}{2}\int_0^z\frac{D_1}{\int D_1\mathrm{d}\varsigma}\mathrm{d}\varsigma+i\frac{T^2}{2\int D_1\mathrm{d}\varsigma}], \quad (12)$$

here

$$u_1=\eta_1\sqrt{\frac{D(z)}{R(z)}}\mathrm{e}^{i\phi_1}\mathrm{sech}\theta_1, \quad (13)$$

where

$$\theta_1=\eta_1t-\eta_1\xi_1\int_0^z D(\varsigma)\mathrm{d}\varsigma-\theta_{10},$$

$$\phi_1=\xi_1t+\frac{1}{2}(\eta_1^2-\xi_1^2)\int_0^z D(\varsigma)\mathrm{d}\varsigma-\phi_{10},$$

$$u_2=\sqrt{\frac{D(z)}{R(z)}}\frac{G}{N}, \quad (14)$$

where

$$\begin{aligned} G&=a_1\cosh\theta_2\mathrm{e}^{i\phi_1}+a_2\cosh\theta_1\mathrm{e}^{i\phi_2} \\ &\quad+ia_3(\sinh\theta_2\mathrm{e}^{i\phi_1}-\sinh\theta_1\mathrm{e}^{i\phi_2}), \\ N&=b_1\cosh(\theta_1+\theta_2)+b_2\cosh(\theta_2-\theta_1) \\ &\quad+b_3\cos(\phi_2-\phi_1), \\ \theta_k&=\eta_kt-\eta_k\xi_k\int_0^z D(\varsigma)\mathrm{d}\varsigma-\theta_{k0}, \\ \phi_k&=\xi_kt+\frac{1}{2}(\eta_k^2-\xi_k^2)\int_0^z D(\varsigma)\mathrm{d}\varsigma-\phi_{k0}, \\ &\quad(k=1,2) \\ a_3&=\eta_1\eta_2(\xi_1-\xi_2), \quad b_3=-\eta_1\eta_2, \\ a_k&=\frac{\eta_k}{2}[\eta_k^2-\eta_{3-k}^2+(\xi_1-\xi_2)^2], \\ b_k&=\frac{1}{4}[(\eta_1+(-1)^k\eta_2)^2+(\xi_1-\xi_2)^2]. \end{aligned} \quad (15)$$

It is interesting to notice that, integrable Eq. (10) can reduce to the one for the NLS with constant coefficient, which can also be obtained by using the Painlevé analysis[15]. On the other hand, since distributed frequency chirp appears in the solutions explicitly, by using exact chirped one- and two-soliton solutions Eqs. (11) and (12), one can consider the evolution process of the soliton with initial frequency chirp in inhomogenous optical fiber media more conveniently and completely. Also, because they include two arbitrary distributed functions $D_1(z)$ and $R_1(z)$, thus by choosing the different form in them, one can explain the various soliton controls with initial frequency chirp. Certainly, one can also obtain the analogeous one- soliton solution by using other method[11].

Here, as an example, we consider an exponential distributed control system with the varying group velocity dispersion parameter

$$D_1(z)=g_1\exp(g_2z), \quad (16)$$

the nonlinearity parameter

$$R_1(z)=r_1\exp(r_2z+g_2z), \quad (17)$$

and the amplification or absorption coefficient

$$\Gamma_1(z)=(g_2+r_2)/2, \quad (18)$$

where $g_1$ and $g_2$ are the parameters used to describe the dispersion, and $r_1$, $r_2$ are related to the nonlinearity. Thus one can rewrite Eqs. (11) and (12) as

$$\begin{aligned} A_1&=\eta_1\sqrt{\frac{g_1}{r_1}\exp(-2g_2z-r_2z)}\mathrm{sech}\theta_1 \\ &\quad\times\exp[i(\phi_1+\frac{g_2}{g_1}\exp(-g_2z)T^2/2)], \\ \theta_1&=\eta_1\exp(-g_2z)(T+\frac{g_1}{g_2}\xi_1)-\theta_{10}, \\ \phi_1&=\exp(-g_2z)[\xi_1T-\frac{1}{2}\frac{g_1}{g_2}(\eta_1^2-\xi_1^2)]-\phi_{10}, \end{aligned} \quad (19)$$

and

$$A_2 = \sqrt{\frac{g_1}{r_1}\exp(-2g_2z - r_2z)}\frac{G}{N} \times \exp[i\frac{g_2}{g_1}\exp(-g_2z)T^2/2], \quad (20)$$

where $G$ and $N$ are given by Eq. (12) with $\theta_k = \eta_k \exp(-g_2z)(T + \frac{g_1}{g_2}\xi_k) - \theta_{k0}$, and $\phi_k = \exp(-g_2z)[\xi_k T - \frac{1}{2}\frac{g_1}{g_2}(\eta_k^2 - \xi_k^2)] - \phi_{k0}$. From the expression of $\theta_k$, we can clearly see that, the inverse pulse width of soliton is dependent on $\eta_k \exp(-g_2z)$, $\frac{g_1}{g_2}\xi_k$ is only a translation quantity in time domain, and $\theta_{k0}$, $\phi_{k0}$ are related to the initial position and initial phase of solitons. For convenience, we take the parameters $g_1 = r_1 = 0.01$, $g_2 = -0.01$. Thus the gain/loss distributed function $\Gamma_1(z) = (g_2 + r_2)/2 = (r_2 - 0.01)/2$, only depends on $r_2$. Figure 1 presents the contour plot of the Eq. (19) for the different values of the parameter $r_2$, respectively, where Figs. 1(a), (b), and (c) correspond to the case with no loss or gain, with the loss, and with the gain, respectively. From it one can clearly see that, chirped soliton can be nonlinearly compressed cleanly and efficiently in an optical fiber. Moreover, under the same initial condition, compression of optical soliton in the optical fiber with the loss is the most dramatic, as shown in Fig. 2. Based on the fact, in the following, our discussion emphasizes on the typical physical situation when an optical fiber is with the loss. Furthermore, it is worth noting that, compared to unchirped soliton, compression effect on the propagation of chirped optical soliton can be seen in Fig. 3. While the soliton without initial chirp is only with the energy decreasing, and without the compression of pulse width.

The evolution behavior of two-soliton solution of Eq. (20) in this situation is shown in Fig. 4. Although the compression is optimal, it is also clear that the area of the pulse is decreasing, i.e., energy is decreasing, which will ultimately limit the amount of compression that can be obtained, that is, compressed soliton may not propagate an arbitrary distance $z$.

Also, it is worth noting that the existence of soliton solutions of Eqs. (19) and (20) depends on the integrable Eq. (18). However, the integrable Eq. (18) may not be satsified in real application. In other words, in reality, the loss of the fiber may not keep constant strictly. Here we take the Eq. (19) as an example, and show the soliton propagation process for nonintegrable system as shown in Fig. 5, in which $\Gamma_1(z)$ takes as $0.005[1 + 0.1\sin(z)]$, compared to $\Gamma_1 = 0.005$ shown in Fig. 1(b). The results show that the main characteristic of evolution for the soliton did not change, that is, the compression is not affected. This may make the compression project more realistic.

To demonstrate stability with respect to finite initial perturbations for chirped soliton, we performed three types of numerical experiments. The results reveal that the finite initial perturbations (10%), such as amplitude, chirp, or white noise could not influence the main character of the solution. Here, we show that the evolution

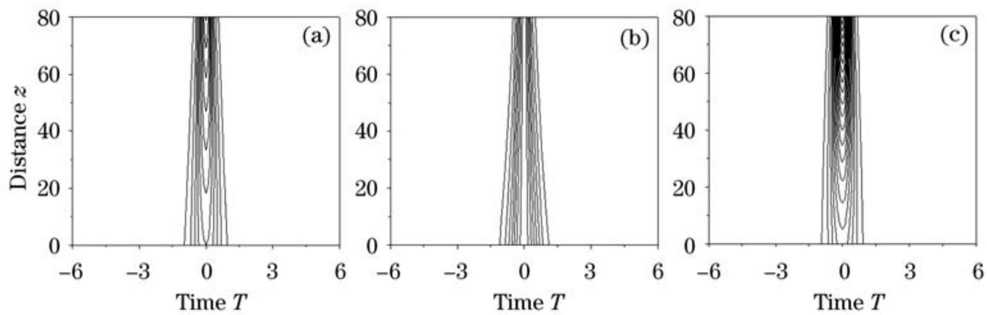


Fig. 1. The contour plot of soliton solution given by Eq. (19), where the parameters adopted are $\eta_1 = 1.5$, $\xi_1 = 0$, $\theta_{10} = 0$, $\phi_{10} = 0$, (a) $r_2 = 0.01$, (b) $r_2 = 0.02$, (c) $r_2 = 0$.

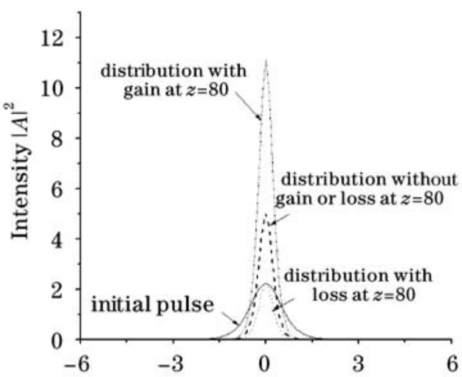


Fig. 2. Comparison of compression of optical soliton with gain, loss, and without gain or loss. The parameters adopted are the same as Fig. 1.

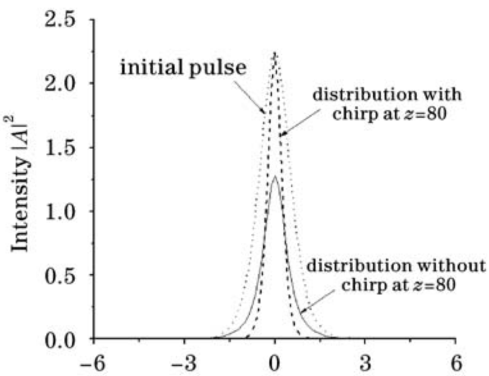


Fig. 3. Comparison of propagation of optical soliton with chirp and without chirp. The parameters adopted are $\eta_1 = 1.5$, $\xi_1 = 0$, $\theta_{10} = 0$, $\phi_{10} = 0$, $r_2 = 0.02$.

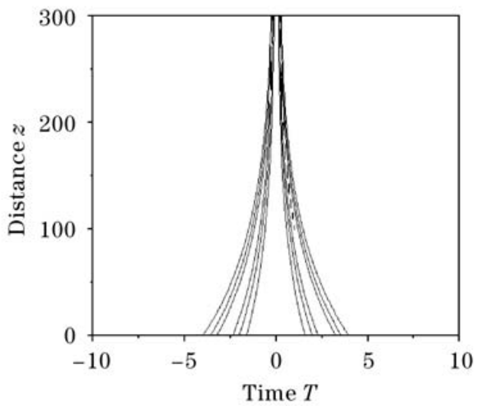


Fig. 4. The contour plot of soliton solution given by Eq. (20), where the parameters adopted are $\eta_1 = 1$, $\eta_2 = -1.1$, $\xi_1 = -0.1$, $\xi_2 = 0.1$, $r_2 = 0.02$, $\theta_{10} = \theta_{20} = 0$, $\phi_{10} = \phi_{20} = 0$.

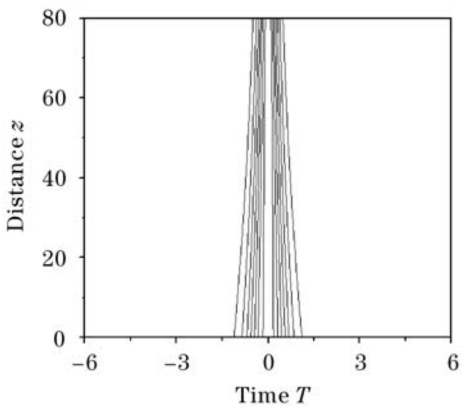


Fig. 5. The contour plot of chirped soliton whose initial shape is given by Eq. (19) in the nonintegrable system, where the parameters adopted are $\eta_1 = 1.5$, $\xi_1 = 0$, $\theta_{10} = 0$, $\phi_{10} = 0$, $\Gamma_1(z) = 0.005[1 + 0.1\sin(z)]$.

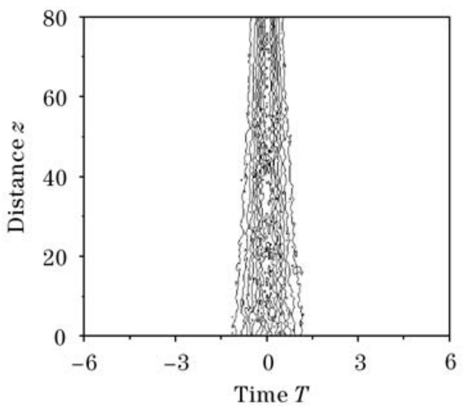


Fig. 6. The contour and evolution plot of chirped soliton whose initial shape is given by Eq. (19) under the perturbation of white noise whose maximal value is 0.1, where the parameters adopted are the same as Fig. 5.

of chirped soliton under the perturbation of white noise whose maximal value is 0.1 as shown in Fig. 6.

In conclusion, we have found exact chirped multi-soliton solutions of the nonlinear Schrödinger equation with varying coefficients. As an example, we have considered the exponential distributed control system, and some main features of solutions have been shown. The results have revealed that chirped soliton can all be nonlinearly compressed cleanly and efficiently in an optical fiber with no loss or gain, with the loss, or with the gain. Furthermore, under the same initial condition, compression of optical soliton in the optical fiber with the loss is the most dramatic. Also, under nonintegrable condition the evolution of soliton has been demonstrated by numerical simulation. Finally, the stability with respect to finite initial perturbations for chirped soliton has been discussed in detail. The results have shown that, the nonintegrable condition, and the finite initial perturbations could not influence the main characteristic of chirped soliton. The application of these results to optical solitons compression in long optical fibers should be an interesting task.

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