Effects of parameters on the stationary state and self-trapping of three coupled Bose-Einstein condensate solitons

Yongshan Cheng (程永山)[1,2], Hong Li (李 宏)[1,2], and Rongzhou Gong (龚荣洲)[1]

[1]*Department of Electronic Science and Technology, Huazhong University of Science and Technology, Wuhan 430074*

[2]*Department of Physics, Hubei Normal University, Huangshi 435002*

Received March 1, 2005

The stability of three coupled Bose-Einstein condensate (BEC) solitons is investigated by the variational approach in two conventional time-independent trapping potentials. The effects of parameters of the potentials and the initial conditions of the BEC soliton system on the stationary state and self-trapping are discussed. It is found that the trapping potentials play an important role in the stability of the system and change the characteristics of the system, and there are different critical potential amplitude values corresponding to different trapping potentials and initial conditions of the BEC soliton system.

OCIS codes: 270.5530, 190.4720, 020.7010.

Discovery of Bose-Einstein condensates (BECs) in dilute atomic alkali-metal gases yields a stunning new demonstration of the wavelike behavior of atoms and provides an important tool for eventual technological applications of BECs[1,2]. The four-wave mixing, the amplification of light and atoms with a condensate, the creation of topological structures such as vortices and vortex lattices, as well as dark and bright solitons are being investigated now. These studies have stimulated a large number of research activities on the nonlinear atom optics[3,4].

A mean-field description for the macroscopic BEC wave-function is constructed using the Hartree-Fock approximation and results in the Gross-Pitaevskii equation (GPE). The quasi one-dimensional (1D) regime is valid when the transverse dimensions of the condensate are of the order of its healing length, and are much smaller than its longitudinal dimension. Thus the condensate has the form of an ellipsoid stretched along one of its major axes. In this regime the BEC remains phase coherent and the GPE reduces to a 1D nonlinear Schrödinger equation (NLSE) with an external potential[5−7]. The dynamics of BECs in the presence of various trapping potentials has been studied in the framework of almost all nonlinear evolution equations possessing quasi-soliton solutions[8−10]. A relevant interesting issue is how to control the motion of nonlinear excitations of the condensates and different types of solitons, including the coupled BEC solitons. Several properties related with solitons in a BEC have been already reported; however, the characteristics of three coupled BEC solitons in trapping potentials and effects of the potentials on the characteristics have not been studied in detail yet. In this letter, the effects of the potentials and the initial conditions of the BEC soliton system on the stationary state and the self-trapping of three coupled BEC solitons are investigated by means of the variational approach, and some novel results are obtained.

In the 1D regime, the three coupled BEC solitons can be described by the following three coupled NLSEs,

$$j\frac{\partial u_1}{\partial t}+\frac{1}{2}\frac{\partial^2 u_1}{\partial z^2}+|u_1|^2u_1=V(z)u_1+Ku_2,$$

$$j\frac{\partial u_2}{\partial t}+\frac{1}{2}\frac{\partial^2 u_2}{\partial z^2}+|u_2|^2u_2=V(z)u_2+K(u_1+u_3),$$

$$j\frac{\partial u_3}{\partial t}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z^2}+|u_3|^2u_3=V(z)u_3+Ku_2, \qquad (1)$$

where u_i ($i=1,2,3$) are the condensate wave-functions, $V(z)$ is the normalized confining potential of the traps in the longitudinal direction (z direction), and K is the linear coupling coefficient arising out of overlaps of the transverse parts of the wave-functions. In this letter, we only consider the case that the coupling between two adjacent condensates is the same and there is no coupling between components 1 and 3. We also notice that, in general, a coupled system includes linear and nonlinear coupling cases[11]. Here, we will be primarily concerned with the linear coupling case.

We adopt the trial functions below as the solution of Eq. (1),

$$u_1(z,t)=\frac{N\cos^4\theta(t)}{\sqrt[4]{\pi}}\exp\left[-\frac{N^2\cos^8\theta(t)}{2}z^2+j\phi(t)+j\varphi\right],$$

$$u_2(z,t)=\frac{N\sin^2 2\theta(t)}{2\sqrt[4]{\pi}}\exp\left[-\frac{N^2\sin^4 2\theta(t)}{8}z^2+j\varphi\right],$$

$$u_3(z,t)=\frac{N\sin^4\theta(t)}{\sqrt[4]{\pi}}\exp\left[-\frac{N^2\sin^8\theta(t)}{2}z^2-j\phi(t)+j\varphi\right], \qquad (2)$$

where $\theta(t)$, φ, and $\phi(t)$ are respectively the coupling angle, the phase, and the phase difference. During the evolution of the three components, u_i retain the Gaussian-shape (quasi-soliton) given by Eq. (2), but θ and ϕ become functions of time. The density of population in a certain component i is $|u_i|^2$ and $N_i(t)=$

$\int_{-\infty}^{\infty}|u_i|^2\mathrm{d}z$ is the number of atoms in each component. $N = N_1 + N_2 + N_3$ is the total number of atoms in the system of three coupled BEC solitons (a conserved quantity).

Two conventional time-independent trapping potentials are considered in this letter. The periodic external potential created along the waveguide axis (z axis) by means of a pair of far-off-resonance laser beams can be described by[12]

$$V_1 = V_0 \sin^2(mz), \quad (3)$$

where V_1 represents the potential caused by the presence of the optical lattice (OL) with the retro-reflected laser beams and m is associated with the OL wave-number, reveals the spatial width of the OL, and can be tuned by the geometry of laser beams.

In the realistic BEC experimental setup, a weak magnetic trap can be simulated by the tanh-shaped potential[13]

$$V_2 = V_0 \tanh^2(nz), \quad (4)$$

where n like m, reveals the spatial width of the tanh-shaped trapping potential. In Eqs. (3) and (4), V_0 is the potential amplitude, which can be either positive or negative.

The averaged Lagrangian of Eq. (1) can be defined with the usual variational approach,

$$\begin{aligned} L(t) = &\int_{-\infty}^{\infty}\left\{\sum_{i=1}^{3}\left[\frac{j}{2}\left(u_i^*\frac{\partial u_i}{\partial t} - u_i\frac{\partial u_i^*}{\partial t}\right) - \frac{1}{2}\left|\frac{\partial u_i}{\partial z}\right|^2 + \frac{1}{2}|u_i|^4 - V|u_i|^2\right] - K(u_1u_2^* + u_3u_2^* + u_2u_1^* + u_2u_3^*)\right\}\mathrm{d}z \\ = &-N\cos 2\theta\frac{\mathrm{d}\phi}{\mathrm{d}t} - 0.05N^3(1 - \frac{3}{4}\sin^2 2\theta)^2 - \frac{0.15}{32}N^3\sin^6 2\theta - \Omega(\theta)V_0 \\ &-\sqrt{2}KN\left[\frac{(1+\cos 2\theta)\sin^2 2\theta}{\sqrt{5 - 6\cos 2\theta + 5\cos^2 2\theta}} + \frac{(1-\cos 2\theta)\sin^2 2\theta}{\sqrt{5 + 6\cos 2\theta + 5\cos^2 2\theta}}\right]\cos\phi, \end{aligned} \quad (5)$$

where the asterisk denotes the complex conjugate and

$$\Omega(\theta) = \begin{cases} \frac{N}{2} - \frac{N}{2}\left[\cos^4\theta\exp\left(\frac{-m^2}{N^2\cos^8\theta}\right) + \frac{1}{2}\sin^2 2\theta\exp\left(\frac{-4m^2}{N^2\sin^4 2\theta}\right) + \sin^4\theta\exp\left(\frac{-m^2}{N^2\sin^8\theta}\right)\right] & \text{for } V_1 \\ N - \frac{N^2\cos^8\theta}{\sqrt{a^2n^2+N^2\cos^8\theta}} - \frac{N^2\sin^4 2\theta}{2\sqrt{4a^2n^2+N^2\sin^4 2\theta}} - \frac{N^2\sin^8\theta}{\sqrt{a^2n^2+N^2\sin^8\theta}} & \text{for } V_2 \end{cases}, \quad (6)$$

where $\tanh^2(z) = 1 - \mathrm{sech}^2(z) \approx 1 - \exp(-a^2z^2)$, and $a = 2\sqrt{\ln 2}/1.76$.

Using $\frac{\partial L(t)}{\partial\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(t)}{\partial\dot{\sigma}}\right) = 0$, $(\sigma = \theta, \phi)$, we can obtain

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \left[\frac{(1+\cos 2\theta)\sin 2\theta}{2\sqrt{5 - 6\cos 2\theta + 5\cos^2 2\theta}} + \frac{(1-\cos 2\theta)\sin 2\theta}{2\sqrt{5 + 6\cos 2\theta + 5\cos^2 2\theta}}\right]\sqrt{2}K\sin\phi, \\ \frac{\mathrm{d}\phi}{\mathrm{d}t} &= -\frac{0.15}{8}N^2\cos 2\theta(5 + 3\cos 4\theta) + \frac{0.45}{16}N^2\sin^4 2\theta\cos 2\theta + \Sigma(\theta)V_0 \\ &\quad+\sqrt{2}K\frac{(1+\cos 2\theta)^3\cos 2\theta - 4\sin^2 2\theta(1-\cos 2\theta)^2}{\sqrt{(5 - 6\cos 2\theta + 5\cos^2 2\theta)^3}}\cos\phi \\ &\quad+\sqrt{2}K\frac{(1-\cos 2\theta)^3\cos 2\theta + 4\sin^2 2\theta(1+\cos 2\theta)^2}{\sqrt{(5 + 6\cos 2\theta + 5\cos^2 2\theta)^3}}\cos\phi, \end{aligned} \quad (7)$$

where

$$\Sigma(\theta) = \begin{cases} -\frac{1}{2}\left[-\left(\cos^2\theta + \frac{2m^2}{N^2\cos^6\theta}\right)\exp\left(\frac{-m^2}{N^2\cos^8\theta}\right) + \left(\cos 2\theta + \frac{8m^2\cos 2\theta}{N^2\sin^4 2\theta}\right)\exp\left(\frac{-4m^2}{N^2\sin^4 2\theta}\right) \right. \\ \quad \left. + \left(\sin^2\theta + \frac{2m^2}{N^2\sin^6\theta}\right)\exp\left(\frac{-m^2}{N^2\sin^8\theta}\right)\right] & \text{for } V_1 \\ -N\left[\frac{-\cos^6\theta(2a^2n^2+N^2\cos^8\theta)}{\sqrt{(a^2n^2+N^2\cos^8\theta)^3}} + \frac{\sin^2 2\theta\cos 2\theta(8a^2n^2+N^2\sin^4 2\theta)}{\sqrt{(4a^2n^2+N^2\sin^4 2\theta)^3}} + \frac{\sin^6\theta(2a^2n^2+N^2\sin^8\theta)}{\sqrt{(a^2n^2+N^2\sin^8\theta)^3}}\right] & \text{for } V_2 \end{cases}, \quad (8)$$

Equation (7) determines the evolution characteristics of the three coupled BEC solitons because Eq. (2) shows that $\theta(t)$ and $\phi(t)$ determine the wave-function of each BEC soliton. On the other hand, Eq. (7) also shows that $\theta(t)$ and $\phi(t)$ are relative to the trapping potential. Hence, the parameters of the trapping potential have effect on the coupled system.

The coupled BEC soliton system is in a stationary state if the time derivatives in Eq. (7) are set to zero. The solitary matter wave persists as a robust structure when a stationary state appears. According to the degenerate GPE eigenstates that break the coupling angle θ symmetry[14−16], from Eq. (7), the critical potential amplitude values corresponding to the stationary state are

$$V_{0c1}(\theta_0,\phi_0)=\Big[\frac{0.15}{8}N^2\cos 2\theta_0(5+3\cos 4\theta_0)-\frac{0.45}{16}N^2\sin^4 2\theta_0\cos 2\theta_0$$
$$\mp\sqrt{2}K\frac{(1+\cos 2\theta_0)^3\cos 2\theta_0-4\sin^2 2\theta_0(1-\cos 2\theta_0)^2}{\sqrt{(5-6\cos 2\theta_0+5\cos^2 2\theta_0)^3}}\mp\sqrt{2}K\frac{(1-\cos 2\theta_0)^3\cos 2\theta_0+4\sin^2 2\theta_0(1+\cos 2\theta_0)^2}{\sqrt{(5+6\cos 2\theta_0+5\cos^2 2\theta_0)^3}}\Big]/\Sigma(\theta_0), \quad (9)$$

where, $\mp$ corresponds to $\phi=0$ and $\phi=\pi$, respectively. We refer to the states as the in-phase case ($\phi_0=0$) and out-of-phase case ($\phi_0=\pi$) solutions.

From Eq. (5), the Hamiltonian $H(\theta,\phi)$ of the three coupled BEC solitons can be obtained. When the initial conditions are selected as in-phase case, there is a pair of eigen-functions of GPE and the ground state energy is $E=1$. If the self-trapping effect takes place, then $H(\theta_0,\phi_0)\geq 1$[14–16], where θ_0 is the initial coupling angle and ϕ_0 is the phase difference. Therefore, when the self-trapping effect takes place, the critical potential amplitude values depend on the following condition

$$V_{0c2}(\theta_0,\phi_0)=\Big[1-0.05N^3(1-\frac{3}{4}\sin^2 2\theta_0)^2$$
$$-\frac{0.15}{32}N^3\sin^6 2\theta_0$$
$$-\sqrt{2}KN\frac{(1+\cos 2\theta_0)\sin^2 2\theta_0}{\sqrt{5-6\cos 2\theta_0+5\cos^2 2\theta_0}}\cos\phi_0$$
$$-\sqrt{2}KN\frac{(1-\cos 2\theta_0)\sin^2 2\theta_0}{\sqrt{5+6\cos 2\theta_0+5\cos^2 2\theta_0}}\cos\phi_0\Big]/\Omega(\theta_0). \quad (10)$$

In Figs. 1 and 2, we plot the critical potential amplitude values V_{0c1} versus θ_0 by numerically solving Eq. (9) for

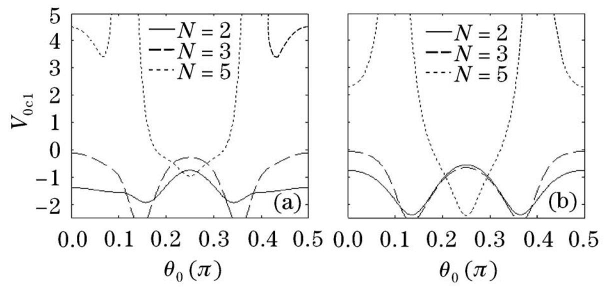

Fig. 1. When the coupled BEC soliton system remains a stationary state ($\phi_0=0$), V_{0c1} versus θ_0 for different trapping potentials V_1 (a) and V_2 (b).

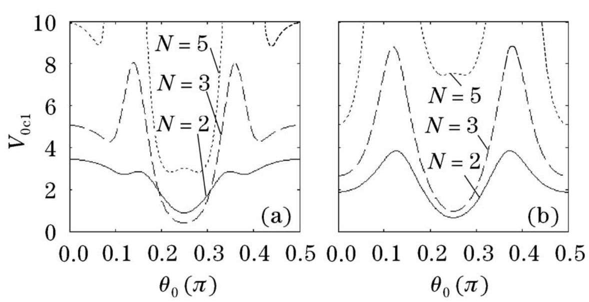

Fig. 2. When the coupled BEC siliton system remains a stationary state ($\phi_0=\pi$), V_{0c1} versus θ_0 for different trapping potentials V_1 (a) and V_2 (b).

different trapping potentials when the coupled BEC soliton system remains a stationary state. θ_0 is considered from 0 to 0.5π because of the periodical characteristics in Eq. (2), $K=1.0$, and $m=n=1$. Figure 3 shows the numerical solution to Eq. (10) with $\phi_0=0$; the same parameters of simulations are selected as those in Figs. 1 and 2. But the solid, dashed, and dotted lines correspond to $N=2,3,4$, respectively.

In the three figures, we find that θ_0, N, and the trapping potentials determine V_{0cl} ($l=1,2$) in either the stationary state or the self-trapping case. V_{0cl} are symmetrical with respect to θ_0 in the two trapping potentials, the symmetrical point is $\theta_0=0.25\pi$ and the reason is the BEC soliton system is symmetrical. From Eq. (2), we can conclude easily that the effects of the switching and self-trapping on the atom population imbalance are like the pendulum bob swing in an oscillation manner. The coupling system is in equilibrium at $\theta_0=0.25\pi$. On the other hand, we notice, according to Eq. (9), V_{0c1} is zero when $\theta_0=0.25\pi$ (This is not indicated in plots because we have avoided the value in our numerical simulations. V_{0c1} will change sharply as θ_0 changes slightly near $\theta_0=0.25\pi$). The number of the atoms in each BEC soliton is $N_1=N/4$, $N_2=N/2$, and $N_3=N/4$ when $\theta_0=0.25\pi$. In this case, the system is totally symmetrical and can remain a stationary state without a trapping potential. Apart from the symmetrical point, there are two maximums or minimums of V_{0c1}. This results from the periodicity of the term of $\sum(\theta)$. $V_0\sum(\theta)$ is the energy arising from the interaction of the trapping potential and the BEC soliton. The conclusion is in agreement with the quantum theory.

As shown in these figures, N plays an important role during the evolution of the BEC soliton. There are three distinguishing curves of V_{0cl} corresponding to different N, and the reason is that N has much effect on the trap ability of a trapping potential. For example, Fig. 3 shows that the bigger N is, the smaller V_{0c2} becomes under the same initial conditions. The self-trapping effects in the coupled BEC result from the nonlinearity of interatomic

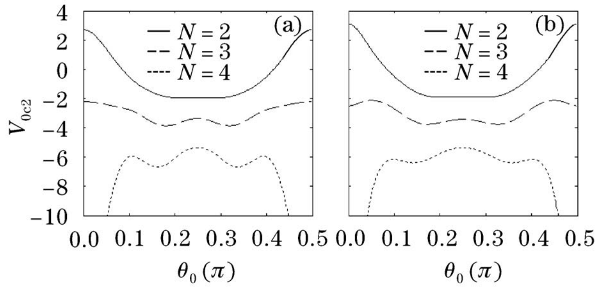

Fig. 3. When the coupled BEC soliton system is self-trapping ($\phi_0=0$), V_{0c2} versus θ_0 for different trapping potentials V_1 (a) and V_2 (b).

interaction in GPE and the long-range quantum coherence of a macroscopic number of atoms, *viz.* the bigger N is, the greater the ability of the self-focusing is.

The different trapping potentials have different abilities to trap the atoms within a BEC soliton system, as shown in Figs. 1 and 2. The trapping ability of V_1 is greater than that of V_2 in the initial in-phase case, but is reverse in the initial out-of-phase case when N is smaller. Meanwhile, Fig. 1 indicates that V_{0c1} is negative. It requires that V_0 in Eqs. (3) and (4) is negative so that the system remains a stationary state. Furthermore, there is a similar change tendency about V_{0cl} when θ_0 becomes larger or smaller both in V_1 and V_2. This is because the shape of one trapping potential around its center is similar to that of the other potential; there are some differences between the shapes of the two potentials near their edges ($V_1 > V_2$). Meanwhile, the BEC soliton systems are the same in the two trapping potentials.

To investigate the effect of the width of a trap on V_{0c2}, in Fig. 4, we plot V_{0c2} versus m and n by numerically solving Eq. (10) for different trapping potentials V_1 and V_2 under the initial in-phase and the self-trapping case. The other parameters are $K = 1.0$, $N = 2$, and $\theta_0 = 0.25\pi$. The effect is strong and V_{0c2} is almost in proportion to the increase of m and n when m and $n < 2$. Subsequently, V_{0c2} remains almost invariable with the increase of m and n. The reason is that the interaction between the trap and the BEC soliton is very strong when the width of the trap is quite narrow (m and n is quite big). The effects of the BEC soliton system on V_{0c2} are dominant. The varying of m and n has smaller effect on the self-trapping. In addition, Fig. 4 also indicates that different trapping potentials have different abilities. V_{0c2} of V_1 is smaller than that of V_2 when m and $n > 0.73$.

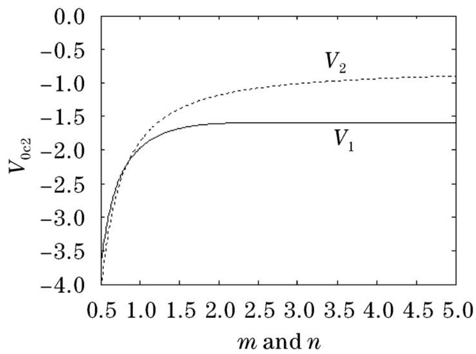

Fig. 4. When the coupled BEC soliton system is self-trapping ($\phi_0 = 0$), V_{0c2} versus m and n (associated with the width of a trapping potential).

Therefore, it can be concluded that the trapping ability of V_1 is greater than that of V_2, which is in agreement with the previous finding.

In a word, these results show that the stationary state and the self-trapping effects depend on the trapping potentials and the initial conditions of the BEC soliton system. These properties can be used to determine the setting in the BEC experiments.

This work was supported by the Science and Technology Program of Education Department of Hubei Province (No. B20052202), the Fund of Natural Science of Hubei Province (No. 2004ABA112), and the Important Program of Education Department of Hubei Province (No. 2002Z00005). Y. Cheng's e-mail address is yong_shan@163.com.

References

1. K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature **417,** 150 (2002).
2. R. Fedele, P. K. Shukla, S. De Nicola, M. A. Man'ko, V. I. Man'ko, and F. S. Cataliotti, JETP Letters **80,** 535 (2004).
3. S. Burger, L. D. Carr, P. Ohberg, K. Sengstock, and A. Sanpera, Phys. Rev. A **65,** 043611 (2002).
4. Y. Wu, X. Yang, and Y. Xiao, Phys. Rev. Lett. **86,** 2200 (2001).
5. G. Huang, J. Szeftel, and S. Zhu, Phys. Rev. A **65,** 053605 (2002).
6. Y. Cheng, H. Li, and D. Yin, Chin. Opt. Lett. **3,** 305 (2005).
7. H. Li, T. Wang, and D. Huang, Chin. Opt. Lett. **2,** 320 (2004).
8. V. S. Shchesnovich, S. B. Cavalcanti, and R. A. Kraenkel, Phys. Rev. A **69,** 033609 (2004).
9. T. Hong, Y. Z. Wang, and Y. S. Huo, Phys. Rev. A **58,** 3128 (1998).
10. V. Ahufinger, A. Sanpera, P. Pedri, L. Santos, and M. Lewenstein, Phys. Rev. A **69,** 053604 (2004).
11. C. Lee, W. Hai, X. Luo, L. Shi, and K. Gao, Phys. Rev. A **68,** 053614 (2003).
12. I. Carusotto, D. Embriaco, and G. C. La Rocca, Phys. Rev. A **65,** 053611 (2002).
13. S.-J. Wang, C.-L. Jia, D. Zhao, H.-G. Luo, and J.-H. An, Phys. Rev. A **68,** 015601 (2003).
14. S. Raghavan and G. P. Agrawal, J. Mod. Opt. **47,** 1155 (2000).
15. B. Deconinck, P. G. Kevrekidis, H. E. Nistazakis, and D. J. Frantzeskakis, Phys. Rev. A **70,** 063605 (2004).
16. H. Sakaguchi and B. A. Malomed, Phys. Rev. E **70,** 066613 (2004).