Photon state-vector function

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The exclusive carrier of photonics is photon, which is a kind of microscopic particles so that obeys the generalized Schrödinger equation, namely the motion equation for a photon. A novel state-vector function that satisfies the equation with three quantum conditions has been constructed, which possesses not only the energy and the momentum but also the angular momentum (spin) for a photon. The analyses of the state-vector function indicate that the macroscopic polarization of light is how to relate with microscopic parameters of a photon such as the probability amplitude and the phase.

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It is well known that the motion of all microscopic particle systems is governed by the generalized Schrödinger equation in quantum mechanics[1]:

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{r}) = \hat{H}\psi(t,\vec{r}), \quad (1)$$

where $\psi(t,\vec{r})$ denotes a complex function describing the quantum state of a particle system, so that it is referred to as the state function[2]. The notation $\hat{H}=\hat{E}$ is the energy operator, which is commonly called the quantum *Hamiltonian*[3]. Equation (1) indicates that the time-derivative operator $i\hbar\partial/\partial t = \hat{H} = \hat{E}$ is equivalent to the energy operator.

The relation between the energy E and the momentum $\vec{p}$ for a photon is $E=\vec{c}\cdot\vec{p}$. Substituting energy operator $\hat{E}=i\hbar\partial/\partial t$ and momentum operator $\hat{\vec{p}}=-i\hbar\vec{\nabla}$ into the relation, and operating on a vector function of $\left|\vec{A}(t,\vec{r})\right\rangle$, the Schrödinger equation for a photon, namely the motion equation for a photon, can be directly written as

$$i\hbar\frac{\partial}{\partial t}\left|\vec{A}(t,\vec{r})\right\rangle = -i\hbar\vec{c}\cdot\vec{\nabla}\left|\vec{A}(t,\vec{r})\right\rangle. \quad (2)$$

The one-dimensional (1D) motion equation for a photon has been given by[4]

$$i\hbar\frac{\partial}{\partial t}\left|\vec{A}(t,z)\right\rangle = -i\hbar c\cdot\frac{\partial}{\partial z}\left|\vec{A}(t,z)\right\rangle. \quad (3)$$

We would name the solutions of Eq. (2) or (3) the photon state-vector functions (SVFs), because they not only describe quantum state for a photon but also possess vector form. It was usually considered as the general solution that is so-called plane wave in quantum mechanics for Eq. (3)

$$\left|\vec{A}(t,z)\right\rangle = \vec{a}\cdot e^{-i(\omega t-\kappa z)}. \quad (4)$$

Besides the SVF (4) satisfies the normalization condition $\left\langle\vec{A}(t,z)\right|\left|\vec{A}(t,z)\right\rangle = 1$, it should satisfy the eigenvalue equations of energy and momentum for a photon, respectively.

$$i\hbar\frac{\partial}{\partial t}\left|\vec{A}(t,z)\right\rangle = \hbar\omega\left|\vec{A}(t,z)\right\rangle, \quad (5)$$

$$-i\hbar\frac{\partial}{\partial z}\left|\vec{A}(t,z)\right\rangle = \hbar\kappa\left|\vec{A}(t,z)\right\rangle. \quad (6)$$

The photon that is described with the SVF (4) possesses the eigenvalues of energy and momentum $E=\hbar\omega$ and $p=\hbar\kappa$, simultaneously. However, owing to lack of description about the angular momentum of a photon, the SVF (4) is incomplete.

We have constructed a novel 1D SVF, which is expressed as

$$\left|\vec{A}(t,z)\right\rangle = \frac{1}{\sqrt{2}}\left[\sigma_+^{(1)}\cdot e^{i\alpha}\begin{pmatrix}1\\ i\end{pmatrix} + \sigma_-^{(1)}\cdot e^{i\beta}\begin{pmatrix}1\\ -i\end{pmatrix}\right]\cdot e^{-i(\omega t-\kappa z)}, \quad (7)$$

here

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\cdot e^{-i(\omega t-\kappa z)} = \left|\vec{A}_+(t,z)\right\rangle \quad (8a)$$

and

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\cdot e^{-i(\omega t-\kappa z)} = \left|\vec{A}_-(t,z)\right\rangle \quad (8b)$$

are a pair of eigen-SVFs for a photon. In addition to the energy eigenvalue $E=\hbar\omega$ and the momentum eigenvalue $p=\hbar\kappa$, the pair of eigen-SVFs (8a) and (8b) possess the eigenvalues $S_{z+}=+\hbar$ and $S_{z-}=-\hbar$ for the spin angular momentum operator $\hat{S}_z=\hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$[5], respectively.

$$\hat{S}_z\left|\vec{A}_+(t,z)\right\rangle = +\hbar\left|\vec{A}_+(t,z)\right\rangle, \quad (9a)$$

$$\hat{S}_z\left|\vec{A}_-(t,z)\right\rangle = -\hbar\left|\vec{A}_-(t,z)\right\rangle. \quad (9b)$$

The eigen-SVFs $\left|\vec{A}_+(t,z)\right\rangle$ and $\left|\vec{A}_-(t,z)\right\rangle$ satisfy the normalization condition $\left\langle\vec{A}_+(t,z)\right|\left|\vec{A}_+(t,z)\right\rangle = \left\langle\vec{A}_-(t,z)\right|\left|\vec{A}_-(t,z)\right\rangle = 1$ respectively, and the orthogonality condition $\left\langle\vec{A}_+(t,z)\right|\left|\vec{A}_-(t,z)\right\rangle = \left\langle\vec{A}_-(t,z)\right|\left|\vec{A}_+(t,z)\right\rangle = 0$ each other. The real part of the eigen-SVF $\left|\vec{A}_+(t,z)\right\rangle$ could describe a wave given by

$$\vec{A}_+(t,z) = \mathrm{Re}\left|\vec{A}_+(t,z)\right\rangle = \mathrm{Re}\left[\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}e^{-i(\omega t-\kappa z)}\right] = \tfrac{1}{\sqrt{2}}\left[\vec{i}\cos(\omega t-\kappa z)+\vec{j}\sin(\omega t-\kappa z)\right]. \quad (10a)$$

Suppose the z-axis is along the propagation direction of light, Eq. (10a) represents the vector with the scalar amplitude of $1/\sqrt{2}$, which is rotating anti-clockwise at an angular frequency ω as seen by an observer who looking back at the light source. According to the stipulation about polarization, such a wave is said to be left circularly polarized (CP) light[6], so that we call $\left|\vec{A}_{+}(t,z)\right\rangle$ the eigen-SVF for left spin photon. In a similar way, we call $\left|\vec{A}_{-}(t,z)\right\rangle$ the eigen-SVF for right spin photon, the real part of which describes right CP light.

$$\begin{aligned}\vec{A}_{-}(t,z) &= \mathrm{Re}\left|\vec{A}_{-}(t,z)\right\rangle \\ &= \tfrac{1}{\sqrt{2}}\left[\vec{i}\cos(\omega t-\kappa z)-\vec{j}\sin(\omega t-\kappa z)\right].\end{aligned} \tag{10b}$$

The general expression of the 1D SVF (7) also satisfies the normalization condition

$$\left\langle\vec{A}(t,z)\right|\left|\vec{A}(t,z)\right\rangle = \left[\sigma_{+}^{(1)}\right]^2+\left[\sigma_{-}^{(1)}\right]^2=1. \tag{11}$$

It is clear from Eq. (11) that the real coefficients $\sigma_{+}^{(1)}$ and $\sigma_{-}^{(1)}$ are the probability amplitudes[7], and α and β are the phases for left and right spin photons, respectively. The expectation value of the spin operator is

$$\begin{aligned}\overline{S_z} &= \left\langle\vec{A}(t,z)\right|\hat{S}_z\left|\vec{A}(t,z)\right\rangle \\ &= \hbar\left[\left(\sigma_{+}^{(1)}\right)^2-\left(\sigma_{-}^{(1)}\right)^2\right].\end{aligned} \tag{12}$$

Just as $\mathrm{Re}\left|\vec{A}_{+}(t,z)\right\rangle$ and $\mathrm{Re}\left|\vec{A}_{-}(t,z)\right\rangle$ describe the waves of left and right CP lights respectively, the wave of general light could be described with the real part of the general SVF, which is given by

$$\begin{aligned}\vec{A}(t,z) &= \mathrm{Re}\left|\vec{A}(t,z)\right\rangle \\ &= \mathrm{Re}\left\{\tfrac{1}{\sqrt{2}}\left[\sigma_{+}^{(1)}\cdot e^{i\alpha}\begin{pmatrix}1\\ i\end{pmatrix}\right.\right. \\ &\qquad\left.\left.+\sigma_{-}^{(1)}\cdot e^{i\beta}\begin{pmatrix}1\\ -i\end{pmatrix}\right]\cdot e^{-i(\omega t-\kappa z)}\right\} \\ &= \tfrac{\vec{i}}{\sqrt{2}}\left[\sigma_{+}^{(1)}\cos(\omega t-\kappa z-\alpha)+\sigma_{-}^{(1)}\cos(\omega t-\kappa z-\beta)\right] \\ &\quad+\tfrac{\vec{j}}{\sqrt{2}}\left[\sigma_{+}^{(1)}\sin(\omega t-\kappa z-\alpha)-\sigma_{-}^{(1)}\sin(\omega t-\kappa z-\beta)\right].\end{aligned} \tag{13a}$$

It is evident that the expectation value is $\overline{S_{z+}}=\hbar$ while $\sigma_{+}^{(1)}=1$ and $\sigma_{-}^{(1)}=0$ on Eq. (12), here all photons remain in the eigenstate of the spin operator with the eigenvalue $+\hbar$, namely all left spin photons. In the meanwhile Eq. (13a) can reduce to

$$\begin{aligned}\vec{A}_{+}(t,z) = \tfrac{1}{\sqrt{2}}&\left[\vec{i}\cos(\omega t-\kappa z-\alpha)\right. \\ &\left.+\vec{j}\sin(\omega t-\kappa z-\alpha)\right].\end{aligned} \tag{13b}$$

It can be seen that Eq. (13b) is the same as Eq. (10a) except for an inessential phase α, both of them describe left CP light. In the case of $\sigma_{+}^{(1)}=0$ and $\sigma_{-}^{(1)}=1$, the expectation value is $\overline{S_{z-}}=-\hbar$ on Eq. (12), and the wave of the light consisting of all right spin photons is given by

$$\begin{aligned}\vec{A}_{-}(t,z) = \tfrac{1}{\sqrt{2}}&\left[\vec{i}\cos(\omega t-\kappa z-\beta)\right. \\ &\left.-\vec{j}\sin(\omega t-\kappa z-\beta)\right],\end{aligned} \tag{13c}$$

which describes right CP light. Supposing $\sigma_{+}^{(1)}=\sigma_{-}^{(1)}=\sqrt{2}/2$, the expectation value is $\overline{S_{z0}}=0$ on Eq. (12), the Eq. (13a) gives out

$$\begin{aligned}\vec{A}_0(t,z) = \mathrm{Re}\left|\vec{A}_0(t,z)\right\rangle &= \cos\left(\omega t-\kappa z-\tfrac{\beta+\alpha}{2}\right) \\ &\cdot\left[\vec{i}\cos\left(\tfrac{\beta-\alpha}{2}\right)+\vec{j}\sin\left(\tfrac{\beta-\alpha}{2}\right)\right].\end{aligned} \tag{13d}$$

If the phase difference $(\beta-\alpha)$ between each pair of left and right spin photons is a constant, Eq. (13d) describes linearly polarized light, the polarization orientation of which makes identical azimuthal angle $\phi=\left(\frac{\beta-\alpha}{2}\right)$; if the distribution of the azimuthal angles is symmetrical in range $(-\pi,\pi)$, Eq. (13d) could describe natural light.

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