Second-order solution of diffusion equation in multiple-scattering media with photon density wave

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The solution of the diffusion equation in multiple-scattering media with photon density wave is investigated and expanded in the theoretical aspect. A second-order analytical solution of diffusion equation in semi-infinite cube space with photon density was researched. Because the flux is the general measure value, and then the relationship between $\delta\mu_a$ and flux J_n is obtained. The relative error between second-order and first-order was presented by the light flux. It is shown that second-order solution is more precise than first-order solution in computation, and the computational precision is more increase especially around the abnormal body. The more accurate theory contributes to the image reconstruction and the quality improvement in imaging.

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The study of light transport through scattering media has recently received increasing attention due to its application to medical diagnosis. In particular, much research is motivated by the ability of photon density wave to image in scattering media[1−9]. Useful information about scattering media such as human tissue can be derived from diffusing photons in the medium. It has been shown that diffusing photons can give information about the average absorption and the scattering properties of the medium as well as detection and characterization of inhomogeneities. Photon density waves make use of modulated intensities with the modulation frequency typically in the radio wave range. The modulated intensity propagates through the scattering medium and exhibits amplitude and phase variations. In experiment, we can measure the optical flux with a CCD camera, get the distribution of $\delta\mu_a$, and then achieve the picture of reconstruction area.

Using the diffusion approximation of the radiative transfer equation, the second-order analytical solution of diffusion equation with photon density wave is presented under the boundary conditions, which is infinite in z direction, and the length and width of cubic being a and b respectively. The solution about the equation in multiple-scattering media with photon density wave is investigated and expanded in the theoretical aspect based on Ref. [6]. In order to hang theory and experiment together, the relationship between $\delta\mu_a$ and flux J_n is figured out. Second-order computation result with a set of optical properties of typical forearm issue is given and compared with the result of first-order solution. It is shown that second-order solution is more precise than first-order solution in computation, and the result can be used in experiment about the image in multiple-scattering media.

The propagation of photons in a scattering medium can be described by the Boltzmann transport equation. In highly scattering media, such as human tissues, the analytic expression for photon propagation can be simplified by a diffusion approximation to the Boltzmann transport equation. The limit of validity for this approximation has been discussed and will not be repeated here. The result time-dependent diffusion equation can be written as

$$\frac{1}{c}\frac{\partial\Phi(\vec{r},t)}{\partial t}+\mu_a(\vec{r})\Phi(\vec{r},t)-\nabla\cdot[D(\vec{r})\nabla\Phi(\vec{r},t)]=S(\vec{r},t), \quad (1)$$

where $\Phi(\vec{r},t)$ is photon density, $\mu_a(\vec{r})$ and $D(\vec{r})$ are the absorption and the diffusion coefficients of the multiple scattering medium, respectively. $S(\vec{r},t)$ is the light source term. We consider the general situation in which the amplitude of the point light source may be modulated at a frequency Ω, i.e., $S(\vec{r},t)=S_0\exp(-i\Omega t)\delta(\vec{r}-\vec{r}_0)$ and $D(\vec{r})=1/\{3[\mu_a(\vec{r})+(1-g)\mu_s(\vec{r})]\}$. $\mu_s(\vec{r})=1/l_s$ is the scattering coefficient with l_s being the scattering mean free path, and $g=<\cos\theta_s>$. c is the speed of light. Based on Ref. [6], density function has form $\Phi(\vec{r},t)=\Phi(\vec{r})e^{-i\Omega t}$, then $\Phi(\vec{r},t)$ can be expressed using the perturbation approach as[4]

$$\Phi(\vec{r})=\Phi_0(\vec{r})+\Phi_1(\vec{r})+\Phi_2(\vec{r}). \quad (2)$$

If the medium is mostly homogenous, it is convenient to express each coefficient as a sum of uniform and non uniform parts:

$$\mu_a(\vec{r})=\mu_a+\delta\mu_a(\vec{r}),$$
$$D(\vec{r})=D+\delta D(\vec{r}). \quad (3)$$

For the experiment model, background multiple scattering media are non-absorption. Let $\mu_a=0$ with $\vec{r}\neq\vec{r}_0$, Eq. (1) changes to

$$\nabla^2\Phi_0+i\frac{\Omega}{Dc}\Phi_0=0, \quad (4)$$

$$\nabla^2\Phi_1+i\frac{\Omega}{Dc}\Phi_1=\frac{\delta\mu_a}{D}\Phi_0, \quad (5)$$

$$\nabla^2\Phi_2+(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D})\Phi_2=\frac{\delta\mu_a}{D}\Phi_1. \quad (6)$$

We assume incident plane wave is from $y=b$ side (see Fig. 1), to solve Eq. (1), we get

$$\Phi_0(x,y)=\frac{4I_0}{\pi}\sum_{n=1,3,5\cdots}\frac{\sin(n\pi x/a)sh\alpha' y}{nsh(\alpha' b)},$$
$$\alpha'=\sqrt{\frac{n^2\pi^2}{a^2}-i\frac{\Omega}{Dc}},$$
$$\Phi_0(x,y=b)=I_0. \quad (7)$$

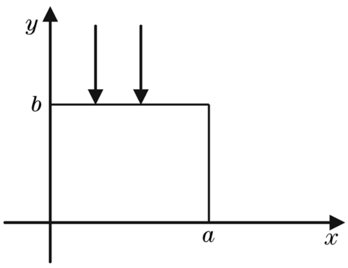

Fig. 1. Illuminance system of experimental arrangement.

The solution of Eq. (5) can be expressed interns of $\Phi_0(\vec{r})$ using the method of Green function, the latter satisfies

$$(\nabla^2 + i\frac{\Omega}{Dc})G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}') = \delta(x-x')\delta(y-y')\delta(z-z'). \quad (8)$$

We assume that Green function is semi-infinite in z direction and the length and width of Green function are a and b respectively. And under boundary condition $G|_{\substack{x=0,x=a\\ y=0,y=b\\ z=0}} = 0$, we get the result of Eq. (8) as

$$G_1(\vec{r},\vec{r}') = -\frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)\sin(\frac{n\pi}{a}x')\sin(\frac{m\pi}{b}y)\sin(\frac{m\pi}{b}y') \times\left\{\begin{array}{c}\frac{\exp[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-i\frac{\Omega}{Dc}}|z-z'|]}{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-i\frac{\Omega}{Dc}}} \\ -\frac{\exp[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-i\frac{\Omega}{Dc}}|z+z'|]}{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-i\frac{\Omega}{Dc}}}\end{array}\right\}. \quad (9)$$

Then Eq. (5) can be rewritten as

$$\Phi_1(\vec{r}) = \int \mathrm{d}\vec{r}' G_1(\vec{r},\vec{r}')\frac{\delta\mu_a(\vec{r}')}{D}\Phi_0(x',y'). \quad (10)$$

Likewise, the solution of Eq. (6) can be expressed as interns of $\Phi_1(\vec{r})$ using the method of Green function, the latter satisfies

$$[\nabla^2 + (i\frac{\Omega}{Dc} - \frac{\delta\mu_a}{D})]G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}') = \delta(x-x')\delta(y-y')\delta(z-z'). \quad (11)$$

Under boundary condition $G|_{\substack{x=0,x=a\\ y=0,y=b\\ z=0}} = 0$, we get the result of Eq. (11) as

$$G_2(\vec{r},\vec{r}') = -\frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{a}x)\sin(\frac{n\pi}{a}x')\sin(\frac{m\pi}{b}y)\sin(\frac{m\pi}{b}y') \times\left\{\begin{array}{c}\frac{\exp[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-\left(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D}\right)}|z-z'|]}{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-\left(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D}\right)}} \\ -\frac{\exp[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-\left(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D}\right)}|z-z'|]}{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2-\left(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D}\right)}}\end{array}\right\}. \quad (12)$$

Then Eq. (6) can be rewritten as

$$\Phi_2(\vec{r}) = \int \mathrm{d}\vec{r}' G_2(\vec{r},\vec{r}')\frac{\delta\mu_a(\vec{r}')}{D}\Phi_1(\vec{r}') \quad (13)$$

At practical work, the physical quantum that we can measure is the optical flux. The relationship between photon density $\Phi(\vec{r})$ and optical flux J_n is

$$J_n = -D\partial\Phi(\vec{r},t)/\partial n. \quad (14)$$

Then we can get the expression forms of optical flux as

$$\begin{aligned} J_n &= -D\left.\frac{\partial\Phi(\vec{r})}{\partial(-y)}\right|_{y=0} \\ &= D\left.\frac{\partial\Phi_2(\vec{r})}{\partial y}\right|_{y=0} + D\left.\frac{\partial\Phi_1(\vec{r})}{\partial y}\right|_{y=0} + D\left.\frac{\partial\Phi_0(x,y)}{\partial y}\right|_{y=0} \\ &= \int \mathrm{d}\vec{r}' \left.\frac{\partial G_2(\vec{r},\vec{r}')}{\partial y}\right|_{y=0}\Phi_1(\vec{r}')\delta\mu_a(\vec{r}') \\ &\quad + \int \mathrm{d}\vec{r}' \left.\frac{\partial G_1(\vec{r},\vec{r}')}{\partial y}\right|_{y=0}\Phi_0(\vec{r}')\delta\mu_a(\vec{r}') \\ &\quad + \frac{D4I_0}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\alpha'}{nsh(\alpha' b)}. \end{aligned} \quad (15)$$

In order to expediently compute numerical value, we use the relationship[10]

$$\begin{aligned} &\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right)\frac{\exp[-\sqrt{\left(\frac{n\pi}{a}\right)^2+k^2}|z-z'|]}{\sqrt{\left(\frac{n\pi}{a}\right)^2+k^2}} \\ &= \frac{a}{2\pi}\times\sum_{n=-\infty}^{\infty}\left\{k_0\left[k\sqrt{(x-x'-2na)^2+(z-z')^2}\right]\right. \\ &\quad \left. -k_0\left[k\sqrt{(x+x'-2na)^2+(z-z')^2}\right]\right\}, \end{aligned} \quad (16)$$

where $k_0(x)$ is the second modified Bessel function of zeroth-order. So the Green function G_1 is reduced to

$$\begin{aligned} G_1(\vec{r},\vec{r}') &= -\frac{1}{\pi a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right) \\ &\sum_{m=-\infty}^{\infty}\{k_0[\alpha'\sqrt{(y-y'-2mb)^2+(z-z')^2}] \\ &-k_0[\alpha'\sqrt{(y+y'-2mb)^2+(z-z')^2}] \\ &-k_0[\alpha'\sqrt{(y-y'-2mb)^2+(z+z')^2}] \\ &+k_0[\alpha'\sqrt{(y+y'-2mb)^2+(z+z')^2}]\}. \end{aligned} \quad (17)$$

So Green function G_2 can be reduced to

$$\begin{aligned} G_2(\vec{r},\vec{r}') &= -\frac{1}{\pi a}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right) \\ &\sum_{m=-\infty}^{\infty}\{k_0[\beta'\sqrt{(y-y'-2mb)^2+(z-z')^2}] \end{aligned}$$

$$-k_0[\beta'\sqrt{(y+y'-2mb)^2+(z-z')^2}]$$
$$-k_0[\beta'\sqrt{(y-y'-2mb)^2+(z+z')^2}]$$
$$+k_0[\beta'\sqrt{(y+y'-2mb)^2+(z+z')^2}]\},$$
$$\beta'=\sqrt{\frac{n^2\pi^2}{a^2}-(i\frac{\Omega}{Dc}-\frac{\delta\mu_a}{D})}. \quad (18)$$

Then the expression forms of optical flux on numerical value is

$$J_n=-\sum_{j,k,l}\Delta x'_j\Delta y'_k\Delta z'_l\frac{1}{\pi a}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right)$$
$$\Big[k_1(\beta'\sqrt{(y'+2mb)^2+(z-z')^2})\bullet\frac{(y'+2mb)\beta'}{\sqrt{(y'+2mb)^2+(z-z')^2}}$$
$$+k_1(\beta'\sqrt{(y'-2mb)^2+(z-z')^2})\bullet\frac{(y'-2mb)\beta'}{\sqrt{(y'-2mb)^2+(z-z')^2}}$$
$$-k_1(\beta'\sqrt{(y'+2mb)^2+(z+z')^2})\bullet\frac{(y'+2mb)\beta'}{\sqrt{(y'+2mb)^2+(z+z')^2}}$$
$$-k_1(\beta'\sqrt{(y'-2mb)^2+(z+z')^2})\bullet\frac{(y'-2mb)\beta'}{\sqrt{(y'-2mb)^2+(z+z')^2}}\Big]$$
$$\times\Phi_1(\vec{r}')\delta\mu_a(\vec{r}')$$
$$-\sum_{j,k,l}\Delta x'_j\Delta y'_k\Delta z'_l\frac{1}{\pi a}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x'\right)$$
$$\Big[k_1(\alpha'\sqrt{(y'+2mb)^2+(z-z')^2})\bullet\frac{(y'+2mb)\alpha'}{\sqrt{(y'+2mb)^2+(z-z')^2}}$$
$$+k_1(\alpha'\sqrt{(y'-2mb)^2+(z-z')^2})\bullet\frac{(y'-2mb)\alpha'}{\sqrt{(y'-2mb)^2+(z-z')^2}}$$
$$-k_1(\alpha'\sqrt{(y'+2mb)^2+(z+z')^2})\bullet\frac{(y'+2mb)\alpha'}{\sqrt{(y'+2mb)^2+(z+z')^2}}$$
$$-k_1(\alpha'\sqrt{(y'-2mb)^2+(z+z')^2})\bullet\frac{(y'-2mb)\alpha'}{\sqrt{(y'-2mb)^2+(z+z')^2}}\Big]$$
$$\times\frac{4I_0}{\pi}\sum_{n'=1,3,5}^{\infty}\frac{\sin\left(\frac{n'\pi}{a}x'\right)sh\left(\alpha'y'\right)}{n'sh(\alpha'b)}\delta\mu_a(\vec{r}')$$
$$+D\frac{4I_0}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\sin\left(\frac{n\pi}{a}x\right)\alpha'}{nsh(\alpha'b)}, \quad (19)$$

where $k_1(r)$ is second modified Bessel function of first-order.

The optical properties[11] were determined to be $\mu_a = 0.23$ cm^{-1}, $\mu_s = 68$ cm^{-1}, relative refractive index $n_r = 1.37$ and $g = 0.9$ using typical forearm tissue. The abnormal body is 0.5 × 0.5 × 0.5 (mm) and placed in the center of the cell, which geometry center at $(x, y, z) = (0.5, 0.5, 0.5)$ (cm). A 120-MHz radio-frequency He-Ne laser provided a parallel beam of 632-nm light incident the plane $y = b$.

The distribution of the relative error between diffusion photon densities Φ_1 and Φ_2 on receive plane is shown in Fig. 2. As expected, the second-order solution of the diffusion equation in semi-infinite cube space is more precise than the first-order solution in computation, and the computational precision is more increase especially around the abnormal body. In actual experiment research, using the optical flux measured in experiment, through special program for data proceeding and calculation, we can get the distribution of $\delta\mu_a$ and then we can get the picture of reconstruction area. From our result, we can say the second-order solution calculating the optical scatter more accurately and it has the actual signification in setting up successfully the reverse arithmetic of imaging with photon density wave. We can also say that it has the theoretical signification in improving the imaging quality and especially improving the distinguish capacity of the abnormal body.

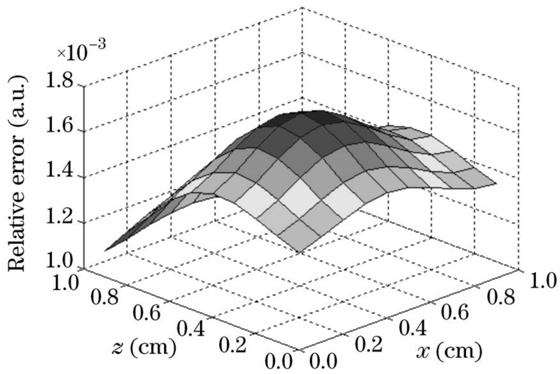

Fig. 2. The distribution of relative error on receive plane.

In this paper we improve the theory of Ref. [6] and work out the relationship between the optical flux J_n and the absorption coefficient $\delta\mu_a$ in multiple-scattering media. Compared with Ref. [6] the results present Green-function and optical flux add a new multinomial, which is the influence of the second-order solution to the photon density function and optical flux. The influence was represented at the added integratiag factor $\Phi_1(\vec{r}')$ in Eq. (19) and the added $\frac{\delta\mu_a}{D}$ of the β in Green-function (12) and optical flux (19). As a consequence, we can say that the result has much theoretic significance on the imaging in multiple-scattering media with photon density wave.

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