

Energy flow in negative index materials

Lorenzo Bolla¹, Michele Midrio¹, and Carlo G. Someda²

¹The Istituto Nazionale per la Fisica della Materia (I. N. F. M.), Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica, Università degli Studi di Udine, viale delle Scienze 208, 33100 Udine, Italy

²I. N. F. M., Dipartimento di Ingegneria dell'Informazione, via G. Gradenigo 6/B, 35131 Padova, Italy

Received April 23, 2004

From Maxwell's equations, we compute the speed and the direction of propagation of active power refracted from air into a negative index material. We prove, both analytically and numerically that the power may refract positively even if phase fronts refract negatively. Considerations on the usage of ray optics in problems involving negative index materials are drawn.

OCIS codes: 260.2110, 080.0080, 250.5300.

Since the experimental demonstration of the feasibility of metamaterials with simultaneously negative permittivity and permeability^[1–3], the phenomenon of negative refraction has attracted an increasing interest, and it has raised fundamental questions on the meaning of the group velocity, and on the relationship between this velocity and the direction of energy flow^[4,5].

Recently, the following statements were proved by Smith, Schurig and Pendry^[6]: as soon as a packet of electromagnetic waves crosses the interface between a positive index material (PIM) (for instance, air), and a negative index material (NIM): 1) a component of the group velocity (in the NIM) follows the interference pattern, but there is also a component *parallel* to the interference wave fronts. Therefore, one cannot easily ascertain the group velocity from the field pattern of the transmitted wave; 2) group and phase velocities both undergo negative refraction.

In this paper, we will confirm the statement 1) by computing analytically the two velocity components. At the same time, we will show that, while statement 2) is certainly correct for a purely harmonic field, a major difference appears as soon as a *modulated* field is concerned. Indeed, as it was numerically observed by Foteinopoulou *et al.*^[7], if one injects into a NIM a wavepacket which is amplitude-modulated by a step function, two very distinct wave motions are observed. The sharp edge of the wavepacket moves along a direction of positive refraction, while the rest of the field, which can be seen as a quasi-harmonic field, refracts negatively.

Therefore, the group velocity, meant to be the velocity of motion of the energy transported by the field, is directed towards positive refraction on the leading edge, and towards negative one on the rest of the beam, which seems to move crabwise, as it was nicely named in Ref. [7].

The aim of the present paper is that of deriving a rigorous expression that allows to analytically describe such a behavior.

Let us consider Maxwell's equations for a field that propagates in a lossless dispersive material in the absence of sources. If we expand the field into a set of plane waves, and retain the only uniform spectrum, since evanescent waves in a semi-infinite medium do not carry any active power, we may write the following relationship for each of the waves $\nabla \times = \mp i\mathbf{k} \times = \mp i\beta(\omega)\hat{\mathbf{k}}(\omega) \times$,

with $\hat{\mathbf{k}}$ the unit vector which is orthogonal to the phase fronts, $\beta = \omega\sqrt{\mu(\omega)\varepsilon(\omega)} > 0$, and sign minus applies to ordinary PIMs, while sign plus applies to NIMs.

We have explicitly written that both β and $\hat{\mathbf{k}}$ may depend on ω . This is what happens when a wavepacket enters a dispersive material, coming from a non-dispersive one, say air: transmitted waves have frequency-dependent phase constants and frequency-dependent directions of the propagation vector.

We now differentiate Maxwell's equations

$$\nabla \times \mathbf{E} = -i\hat{\mathbf{k}}(\omega)\beta(\omega) \times \mathbf{E} = -i\omega\mu\mathbf{H}, \quad (1)$$

$$\nabla \times \mathbf{H} = -i\hat{\mathbf{k}}(\omega)\beta(\omega) \times \mathbf{H} = -i\omega\varepsilon\mathbf{E}, \quad (2)$$

with respect to ω , vector multiply Eq. (1) the complex conjugate of \mathbf{H} , Eq. (2) the complex conjugate of \mathbf{E} and finally subtract the obtained second relationship from the first. Simple algebraic computations yield

$$\begin{aligned} & \text{Re} \left\{ \mathbf{P} \cdot \left[\pm \left(\mathbf{k} + \frac{\beta(\omega)}{\partial\beta/\partial\omega} \frac{\partial\mathbf{k}}{\partial\omega} \right) \right] \right\} \\ & = \frac{1}{\partial\beta/\partial\omega} \left[\frac{\partial(\mu\omega)}{\partial\omega} \frac{|\mathbf{H}|^2}{4} + \frac{\partial(\omega\varepsilon)}{\partial\omega} \frac{|\mathbf{E}|^2}{4} \right], \end{aligned} \quad (3)$$

where sign plus and minus apply to PIMs and NIMs, respectively, and $\mathbf{P} = \mathbf{E} \times \mathbf{H}^*/2$ is the Poynting vector.

Letting

$$\mathbf{u} = \pm \left(\mathbf{k} + \frac{\beta(\omega)}{\partial\beta/\partial\omega} \frac{\partial\mathbf{k}}{\partial\omega} \right) = \pm(\mathbf{k} + A_{\perp}\mathbf{k}_{\perp}), \quad (4)$$

Eq. (3) can be recast as

$$\text{Re} \{ \mathbf{P} \cdot \mathbf{u} \} = v_g \left[\frac{\partial(\mu\omega)}{\partial\omega} \frac{|\mathbf{H}|^2}{4} + \frac{\partial(\omega\varepsilon)}{\partial\omega} \frac{|\mathbf{E}|^2}{4} \right], \quad (5)$$

with $v_g^{-1} = \partial\beta/\partial\omega$. The physical meaning of Eq. (5) is the following: the real part of the Poynting vector flux through a unit-area surface, orthogonal to $\hat{\mathbf{u}}$, equals a term, v_g , whose dimensions are those of a velocity, times the energy density of the electromagnetic field. As in fluid dynamics, this tells us that the energy moves with the velocity v_g , along the direction given by the unit vector $\mathbf{u}/|\mathbf{u}|$, that we call the energy propagation vector.

Notice that \mathbf{u} contains a contribution which is proportional to the derivative of the unit vector $\hat{\mathbf{k}}$. This contribution is orthogonal to $\hat{\mathbf{k}}$, in agreement with findings

of Ref. [6]. Notice that the magnitude of the component orthogonal to $\hat{\mathbf{k}}$ depends on $\beta(\omega)$, and it tends to vanish when the refractive index approaches the null value. In this case the energy propagation vector tends to become antiparallel to the phase fronts.

Two checkpoints are also worth of notice:

- for normally impinging waves $\partial\hat{\mathbf{k}}/\partial\omega = 0$ and $\mathbf{u} \equiv \pm\mathbf{k}$, the group velocity does not have any component orthogonal to the phase vector, consistent with Smith *et al.*[6]. Moreover, we see that in a NIM the active power moves in the opposite sense, compared to the phase;

- if the NIM were not dispersive, i.e. if $\partial\mathbf{k}/\partial\omega = 0$ even for non-orthogonal incidence, the energy would move parallel to $-\hat{\mathbf{k}}$, eventually leading to superlensing. This result is consistent with that observed by Ziolkowski and Heyman[8], where it was numerically verified that focusing of electromagnetic waves in a virtually sizeless point may be achieved in an ideally non-dispersive NIM only.

So far, statement 1) was reaffirmed: the direction of motion of the electromagnetic field energy in a dispersive medium contains both a component which is parallel to the phase fronts, and one which is orthogonal to them. Therefore, the path followed by the energy may not be immediately grasped if the interference pattern of a given wave packet is looked at.

We now pass to consider statement 2), and show that if we explicitly compute the magnitude of the component which is orthogonal to $\hat{\mathbf{k}}$, we may prove that, as noticed by Foteinopoulou *et al.*, two very different behaviors may be observed between a harmonic field, and an amplitude-modulated one[7].

Let us begin with the case of a linearly polarized harmonic field which propagates through the same interface used by Pendry[4]: we assume that the field propagates from air into a Drude medium having the following permeability and permittivity $\varepsilon(\omega)/\varepsilon_0 = \mu(\omega)/\mu_0 = 1 - \omega_p^2/\omega^2$. Moreover, we also assume that the wave frequency is $\omega = \omega_p/\sqrt{2}$, so that the NIM index is equal to -1 . At steady state, the electric and magnetic fields of the wave in the NIM and the wavevector \mathbf{k}_0 form a left-handed set of three mutually orthogonal vectors. Subscript "0" is used here and in the rest of the paper to identify the wavevector at the frequency $\omega = \omega_p/\sqrt{2}$.

Given the direction of the electric and magnetic field, the Poynting vector turns out to be antiparallel to \mathbf{k}_0 . Therefore, when the inner product between \mathbf{u} and \mathbf{P} is computed in Eq. (5), the component of vector \mathbf{u} which is parallel to $\hat{\mathbf{k}}_{0,\perp}$ gives no contribution. Energy propagates along the direction of negative refraction given by the unit vector $-\hat{\mathbf{k}}_0$.

This behavior is clearly illustrated by Fig. 1, where we plot the field distribution that we obtain by means of an FDTD simulation when we inject a beam from air into the Drude medium, and we let the simulation run for a time long enough to reach steady state. In the figure, we also plot the "rays", defined as those lines that are orthogonal, in any point, to the phase fronts. The transient behavior of the field will be illustrated below, where we will discuss our Fig. 3.

Let us come to the case of an amplitude-modulated

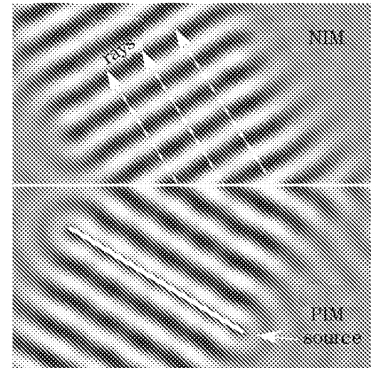


Fig. 1. Steady-state distribution of the field after refraction at the interface between air and the NIM.

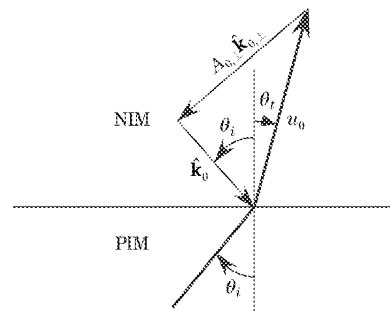


Fig. 2. Refraction of energy at the interface between a PIM (for instance, air) and a dispersive NIM.

beam. Owing to the amplitude modulation, the beam spectrum is composed by a continuum of frequencies, and each frequency refracts towards a different direction because of the dispersive nature of the NIM. In the case of a beam modulated by a sharp step function, this fact is particularly important on the leading edge, where frequencies that are very far away from the carrier one are present.

Differently from the steady-state case, the Poynting vector is no longer antiparallel to $-\hat{\mathbf{k}}_0$. In other words, there are field components, precisely those give rise to the leading edge, for which the inner product between the Poynting vector and $\hat{\mathbf{k}}_{0,\perp}$ no longer vanishes. Therefore, these components travel along a different direction from the steady-state ones.

Let us quantify the matter in the case of the Drude medium we introduce above. The energy propagation vector is easily found to be

$$\mathbf{u}_0 = -(\mathbf{k}_0 + A_{0,\perp}\mathbf{k}_{0,\perp}) = -\left(\mathbf{k}_0 + \frac{4}{3}\tan(\theta_i)\mathbf{k}_{0,\perp}\right), \quad (6)$$

with θ_i the angle of incidence (see Fig. 2). The angle of refraction for the energy carried by the non-steady-state components is

$$\theta_t = \arctan\left(\frac{4}{3}\tan(\theta_i)\right) - \theta_i > 0. \quad (7)$$

Incidentally, notice that this angle is always greater than zero since $\arctan(A_{\perp}) > \theta_i$ for any θ_i . Therefore, the energy refracts positively, in agreement with numerical results obtained in Ref. [7].

This fact is further confirmed by the simulations we illustrate in Fig. 3, where we show two snapshots we obtained during the build-up of the field whose steady state was reported in Fig. 1. As it may be seen, the leading edge of the amplitude modulation refracts into the NIM along a different path from the rays.

The direction of energy refraction of the sharp leading edge is given by Eqs. (4) and (5), and in the specific case of Fig. 3, where the angle of incidence was set equal to $\theta_i \simeq 35^\circ$, it turned out to be $\theta_t \simeq 8^\circ$. Theoretical direction of vector \mathbf{u}_0 is displayed in Fig. 3 with a dashed line. As it may be seen, it nicely matches with the path of energy motion that is obtained numerically.

Note that the computation of the direction of energy flow is not simply a matter of refractive index. Indeed, one might erroneously think that, since the leading edge of the modulated beam contains frequencies that extend up to infinity, and since at these very-high frequencies the refractive index of the Drude medium equals 1, the sharp modulation travels as if no interface is present. As it is proved by Fig. 3, this is not the case, and the correct way of computing the direction of propagation of the energy in the leading edge is making use of Eqs. (4) and (5) above.

We have focused on the propagation of electromagnetic energy through the interface between air and a dispersive material (even with negative index). We have shown, both numerically and analytically, that in the dispersive material the group velocity has both a component that is parallel to the phase fronts, and one that is orthogonal to them. At the interface between air and a NIM, the two components add so that the energy refracts positively on the sharp edges of a modulated beam, while it refracts negatively at the steady state.

Basically, the energy motion in air to NIM material may be schematically described as follows. Let us consider a wavepacket with finite spatial extension, and trace

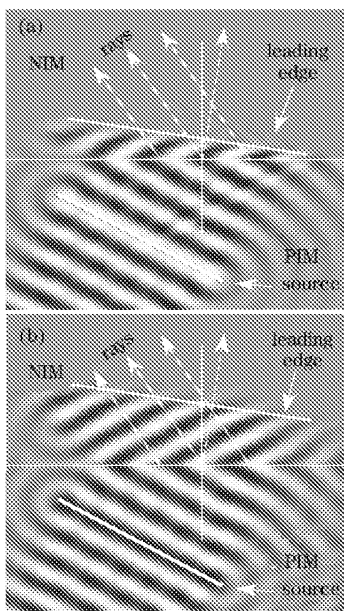


Fig. 3. Build-up of the amplitude-modulated beam that enters the NIM from air, as shown in two different time instants. The leading edge refracts along a different path with respect to the rays (see also Fig. 2).

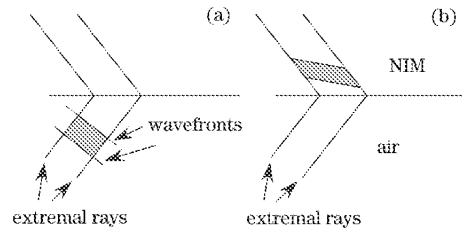


Fig. 4. Schematic illustration of refraction in an air-to-NIM interface. The dashed region represents an time-amplitude-modulated wavepacket with finite spatial extension. (a): The wavepacket is injected from air into the NIM. Its amplitude modulation fronts are orthogonal to the direction of propagation. (b): After refraction, the sharp leading and trailing edges are no longer orthogonal to the direction of propagation.

the rays, i.e. those geometrical lines that are orthogonal, in any point, to the phase fronts. In particular, trace the extremal rays, i.e. the two rays that enclose the wavepacket (see Fig. 4(a)). At the interface with the NIM, the rays bend in agreement with Snell's law of refraction, thus partitioning the whole space into three regions. The first is the region between the extremal rays, the remaining two are the regions outside them. As we have shown above, the energy of the electromagnetic wave stays confined within the first region, i.e. within the extremal rays. Nonetheless, if the wavepacket is time-modulated, i.e. it is not a purely harmonic field, a distortion of the wave fronts is observed. In "ordinary" materials (like air), the wavefronts are orthogonal to the rays. Whereas, this is no longer true in NIMs. Figure 4(b) gives an intuitive description of this phenomenon.

As a side finding of the paper, we also notice that ray optics should be used with some care when propagation into materials with very large dispersion is concerned. The reason is simple: Ray optics is customarily used under the assumption that the direction of a ray, i.e. the direction which is orthogonal to the phase fronts, coincides with the direction of motion of the energy that is carried by a wave. As we have shown, this is not necessarily true for wavepackets that propagate into dispersive media, so that ray optics does not give an immediate picture of the energy motion in this case.

M. Midrio is the author to whom the correspondence should be addressed, his e-mail address is midrio@uniud.it.

References

1. V. G. Veselago, *Sov. Phys. Usp.* **10**, 509 (1968).
2. D. R. Smith, W. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, *Phys. Rev. Lett.* **84**, 4184 (2000).
3. R. A. Shelby, D. R. Smith, and S. Schultz, *Science* **292**, 77 (2001).
4. J. B. Pendry, *Phys. Rev. Lett.* **85**, 3966 (2000).
5. P. M. Valanju, R. M. Walser, and A. P. Valanju, *Phys. Rev. Lett.* **88**, 187401 (2002).
6. D. R. Smith, D. Schurig, and J. B. Pendry, *Appl. Phys. Lett.* **81**, 2713 (2002).
7. S. Foteinopoulou, E. N. Economou, and C. M. Soukoulis, *Phys. Rev. Lett.* **90**, 107402 (2003).
8. R. W. Ziolkowski and E. Heyman. *Phys. Rev. E* **64**, 056625 (2001).