Studies of beam propagation characteristics on apertured fractional Fourier transforming systems

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Based on the principle that a rectangular function can be expanded into a sum of complex Gaussian functions with finite numbers, propagation characteristics of a Gaussian beam or a plane wave passing through apertured fractional Fourier transforming systems are analyzed and corresponding analytical formulae are obtained. Analytical formulae in different fractional orders are numerically simulated and compared with the diffraction integral formulae, the applicable range and exactness of analytical formulae are confirmed. It is shown that the calculating speed of using the obtained approximate analytical formulae, is several hundred times faster than that of using diffraction integral directly. Meanwhile, by using analytical formulae the effect of different aperture sizes on Gaussian beam propagation characteristics is numerically simulated, it is shown that the diffraction effect can be neglected when the aperture size is 5 times larger than the beam waist size.

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In 1993, the fractional Fourier transform (FRFT) was firstly introduced to optics[1−3]. It has become an advanced and international special subject in optics and has been extensively applied in many aspects[4−10]. Optical signal processing and beam control by using FRFT are no longer limited by space, compared with that by using the Fourier transform. Observing planes of arbitrary fractional order can be chosen freely according to practical need. A lot of work on space filtering, beam shaping, and phase recovery has been studied[11−13]. In practice, apertured optical systems often exist, being natural or induced on purpose. So studies of beam propagation characteristics on apertured fractional Fourier transforming systems are necessary.

From the optical point of view there exist two different approaches to obtaining definitions of a FRFT, one is based on Hermite-Gaussian modes and can be implemented optically by means of quadratic graded index (GRIN) media[1], the other is based on rotating the Wigner distribution function of the input signal by a certain angle and can be implemented by means of bulk optical setups[2]. Because quadratic GRIN media have very narrow spatial bandwidth product, the amount of information that GRIN media can deal with is small and the volume of it is very large, the FRFT is implemented by means of bulk optical setups. The two optical setups for implementing the FRFT suggested by Lohmann are shown in Fig. 1. Lohmann introduced two parameters Q and R to determine these two setups[2]

$$f = f_s/Q, \quad d = f_s R, \tag{1}$$

where f_s is the standard focal length, f is the focal length of lens, d is, for the type I Lohmann system, the distance between the input plane and the lens plane, or the distance between the lens plane and the output plane, and for the type II Lohmann system, d is the distance between two lenses. The two parameters relate to the order of FRFT, for type I

$$R = \tan(\phi/2), \quad Q = \sin\phi, \tag{2}$$

for type II

$$R = \sin\phi, \quad Q = \tan(\phi/2), \tag{3}$$

where $\phi = p\pi/2$, p is the order of FRFT.

Generally, the definition of a FRFT of order p for a given input signal $u(x_0)$ in the case of bulk optics is given by

$$u_p(x) = \Im^p[u(x_0)] = C_1 \int_{-\infty}^{\infty} u(x_0) \exp\left(i\pi \frac{x_0^2 + x^2}{\lambda f_s \tan\phi}\right) \times \exp\left(-i2\pi \frac{x x_0}{\lambda f_s \sin\phi}\right) \mathrm{d}x_0, \tag{4}$$

where

$$C_1 = \frac{\exp\left[-i\left(\frac{1}{4}\pi \mathrm{sgn}(\sin\phi) - \frac{1}{2}\phi\right)\right]}{|\lambda f_s \sin\phi|^{\frac{1}{2}}}, \tag{5}$$

and λ is the optical wavelength.

Equation (4) describes an ideal FRFT. In practice,

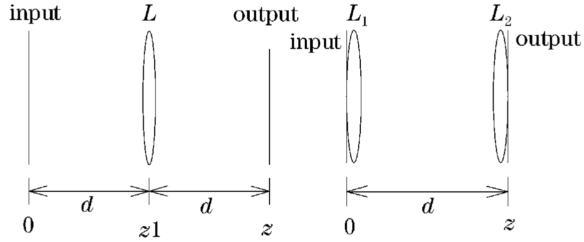

Fig. 1. Two optical setups for performing the FRFT. (a) Type I; (b) type II.

lenses are of finite sizes in fractional Fourier transforming systems. In this letter, we just consider the case when lenses are apertured. In one-dimensional (1D) case, Collins formula[14,15] is given as

$$E(x_2,z)=\sqrt{\frac{i}{\lambda B}}\int_{-a}^{a} \times E(x_1,0)\exp\left[-\frac{i\pi}{\lambda B}\left(Ax_1^2-2x_1x_2+Dx_2^2\right)\right]\mathrm{d}x_1, \quad (6)$$

where the constant phase factor is omitted. In paraxial approximation, matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ can describe any linear optical system. a is the half-width of the lens aperture.

Introducing the hard-edge aperture function $A_p(x)=\begin{cases}1, & |x|\le a \\ 0, & |x|>a\end{cases}$, Eq. (6) can be rewritten as

$$E(x_2,z)=\sqrt{\frac{i}{\lambda B}}\int_{-\infty}^{\infty} E(x_1,0)A_p(x_1) \times\exp\left[-\frac{i\pi}{\lambda B}\left(Ax_1^2-2x_1x_2+Dx_2^2\right)\right]\mathrm{d}x_1. \quad (7)$$

Generally, the hard-edge aperture function can be expanded as the sum of complex Gaussian functions with finite numbers and is given by

$$A_p(x)=\sum_{n=1}^{N}A_n\exp\left(-\frac{B_n}{a^2}x^2\right), \quad (8)$$

where A_n and B_n are the expansion and Gaussian coefficients, respectively, which could be obtained by optimization calculation directly[16,17].

In the case of type I Lohmann system shown in Fig. 1(a), because of the apertured lens, the whole system can be separated into two parts, the first is the free propagation from the input plane to the z_1 plane, and the second is the apertured propagation from the z_1 plane to the output plane because of the finite size of lens. For the first one we have

$$E(x,z_1)=\sqrt{\frac{i}{\lambda d}}\int_{-\infty}^{\infty} E(x_1,0) \times\exp\left[-\frac{i\pi}{\lambda d}\left(x_1^2-2x_1x+x^2\right)\right]\mathrm{d}x_1, \quad (9)$$

and for the second one we have

$$E(x_2,z)=\sqrt{\frac{i}{\lambda d}}\int_{-a}^{a} E(x,z_1) \times\exp\left\{-\frac{i\pi}{\lambda d}\left[\left(1-\frac{d}{f}\right)x^2-2xx_2+x_2^2\right]\right\}\mathrm{d}x. \quad (10)$$

The constant factor $\sqrt{i/(\lambda d)}$ is omitted in the following numerical calculation, as it only determines the absolute intensity and has no influence on the output field distribution. Suppose that a Gaussian beam represented by $\exp(-x^2/w_0^2)$ is just located in the input plane of type I Lohmann system. Inserting Eq. (9) into (10), using Eqs. (7) and (8) and the integral formula

$$\int_{-\infty}^{\infty}\exp\left(-Ax^2\pm Bx\right)\mathrm{d}x=\sqrt{\frac{\pi}{A}}\exp\left(\frac{B^2}{4A}\right), \quad (11)$$

the approximate analytical expression of output field distribution of a Gaussian beam passing through the type I Lohmann system is obtained as

$$E(x_2,z)=\sum_{n=1}^{N}A_n\sqrt{\frac{1}{C_n}} \times\exp\left[-\left(\frac{i\pi}{\lambda d}+\frac{\pi^2}{\lambda^2d^2C_n}\right)x_2^2\right] \quad (12)$$

and

$$C_n=\frac{B_n}{a^2}+\frac{i\pi}{\lambda d}+\frac{\pi^2}{\lambda^2d^2\left(\frac{1}{W_0^2}+\frac{i\pi}{\lambda d}\right)}+\frac{i\pi}{\lambda d}\left(1-\frac{d}{f}\right), \quad (13)$$

where W_0 is the waist width of Gaussian beam.

Similarly, the approximate analytical expression of output field distribution of a plane wave passing through the type I Lohmann system can be obtained and is same as Eq. (12) in form except for

$$C_n=\frac{B_n}{a^2}+\frac{i\pi}{\lambda d}\left(1-\frac{d}{f}\right). \quad (14)$$

In the case of type II Lohmann system shown in Fig. 1(b), there exist two apertures on the respective positions of the lenses; in fact, the second aperture only truncates the output field distribution. According to Collins formula and using the similar approach as for type I, the approximate analytical expression for the output field distribution of a Gaussian beam in the fractional Fourier transforming plane of type II is derived as

$$E(x_2,z)=\sum_{n=1}^{N}A_n\sqrt{\frac{1}{C_n}} \times\exp\left\{-\left[\frac{i\pi}{\lambda d}\left(1-\frac{f}{d}\right)+\frac{\pi^2}{\lambda^2d^2C_n}\right]x_2^2\right\} \quad (15)$$

and

$$C_n=\frac{B_n}{a^2}+\frac{1}{W_0^2}+\frac{i\pi}{\lambda d}\left(1-\frac{d}{f}\right). \quad (16)$$

Similarly, the approximate analytical expression of output field distribution of a plane wave passing through the type II Lohmann system can be obtained and is the same as Eq. (15) in form except for

$$C_n=\frac{B_n}{a^2}+\frac{i\pi}{\lambda d}\left(1-\frac{d}{f}\right). \quad (17)$$

In the following numerical calculation, the expansion coefficient A_n and Gaussian coefficient B_n are given in Ref. [16], where $N=10$. Figures 2 and 3 show the relative intensity logarithm distribution in the output plane for different fractional orders when a Gaussian beam or a plane wave passes the apertured fractional Fourier transforming systems, respectively, where we choose $W_0 : a = 1$. In order to see the detailed differences between the two methods, logarithm distribution is used. Figures 2 and 3 give the same conclusions about these two different calculating methods whether a Gaussian beam or a plane wave is incident. This also proves following results to be reliable. In the case of fractional order $p = 1.0$, results obtained by these two different ways are the same for the two type Lohmann systems, so numerical result calculated by using the approximate analytical expression instead of diffraction integral is appropriate in the case of Fourier transform. In the case of fractional order $p = 0.5$, results obtained by these two different ways are almost the same except for some difference at the edge. In general, difference in results obtained by using these two calculating methods becomes larger when the fractional order becomes smaller for the two type Lohmann systems. The difference in results obtained by these two ways for the type II Lohmann system is weaker than that for the type I Lohmann system. The centers of lines keep well with each other. It can be seen from Figs. 2 and 3 that results obtained by these two ways keep well with each other in the case of $p = 0.2$ for the type I Lohmann system, but for $p = 0.1$ the difference is large. It shows that numerical result calculated by the approximate analytical expression cannot be used to take the place of

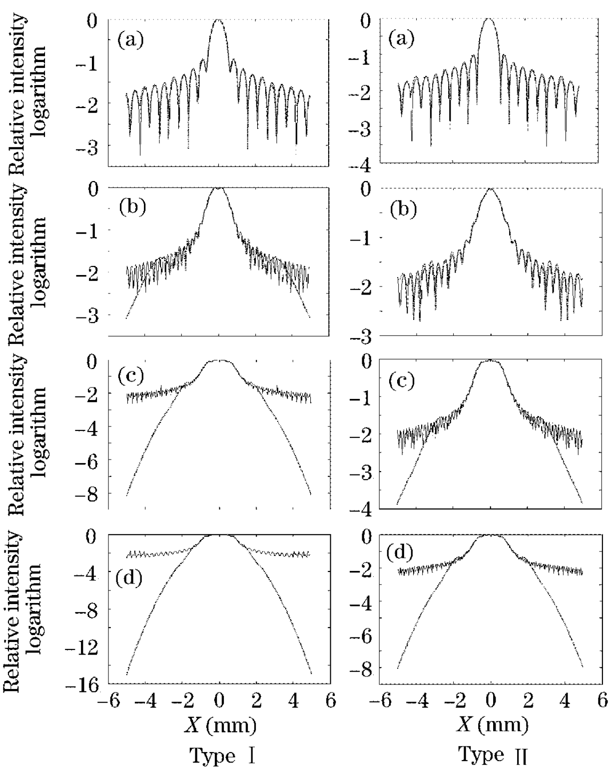

Fig. 2. Relative intensity logarithm distribution in the output plane of the type I and type II systems using different calculating methods for different fractional orders with the incidence of a Gaussian beam. The solid lines denote the case by using the diffraction integral formula, and the dotted lines mean the case by using analytical formula. (a) $p = 1$; (b) $p = 0.5$; (c) $p = 0.2$; (d) $p = 0.1$.

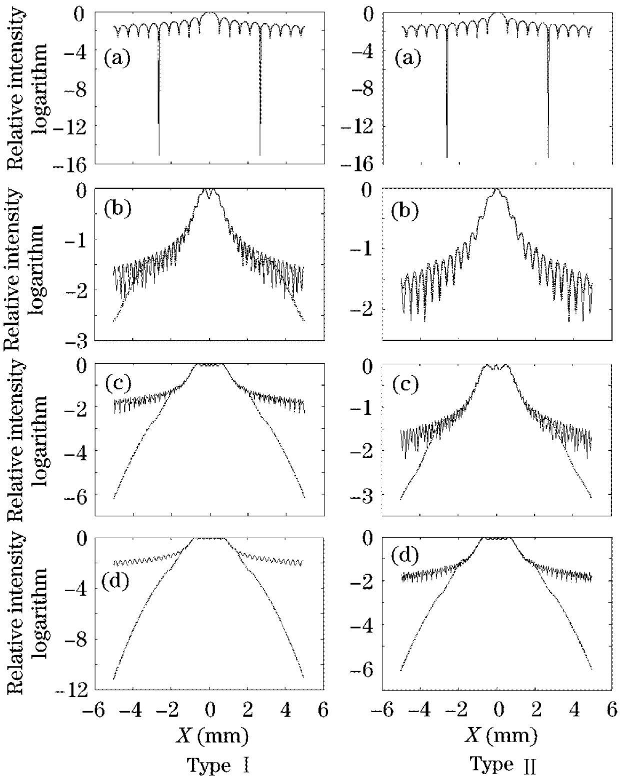

Fig. 3. Relative intensity logarithm distribution in the output plane of the type I and type II systems using different calculating methods for different fractional orders with the incidence of a plane wave. The solid lines denote the case by using the diffraction integral formula, and the dotted lines mean the case by using analytical formula. (a) $p = 1$; (b) $p = 0.5$; (c) $p = 0.2$; (d) $p = 0.1$.

that calculated by the diffraction integral formula in the case of fractional order $p = 0.1$ for the type I Lohmann system. For the type II Lohmann system, in the case of $p = 0.1$, results obtained by these two calculating methods keep well with each other. It can be seen that the method of calculating the diffraction field by using the analytical expression based on the fact that a rectangular function can be expanded into a sum of complex Gaussian functions with finite numbers is appropriate in the case of $p >= 0.2$ for the two type Lohmann systems. It should be pointed out that the analytical expressions in this letter are obtained in the case of $N = 10$. In order to obtain a more precise analytical expression, we should choose a larger N.

It is worth pointing out that numerical calculation for the diffraction integral is time-consuming. Calculating the diffraction integral one time, two or three minutes are spent when 300 points are sampled. Calculating the analytical expression one time, only one or two seconds are spent when 1000 points are sampled. Running speed is increased by several hundred times. The superiority of calculation by using the analytical expression is obvious.

In the case of fractional order $p = 0.5$, results obtained by the analytical expression and by the diffraction integral are almost the same and running speed of the former is far faster than that of the latter. So the relative intensity logarithm distribution in the output plane

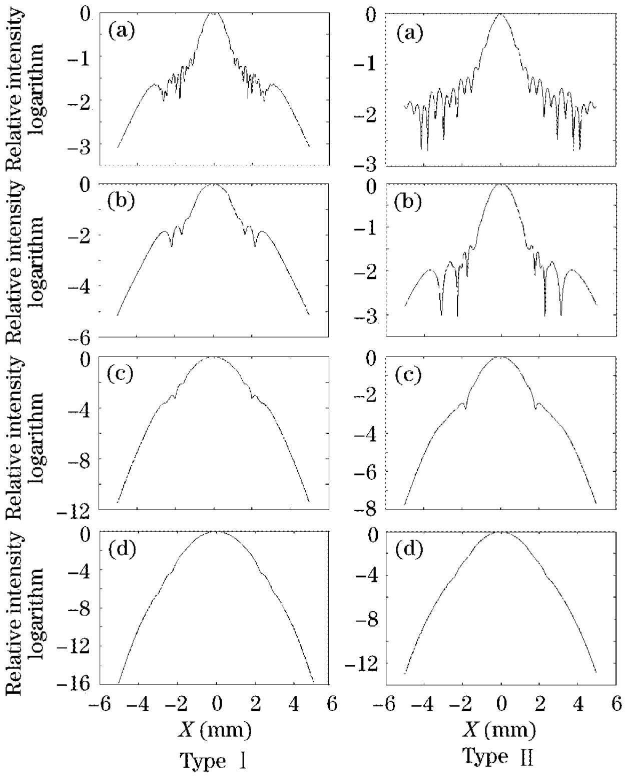

Fig. 4. Relative intensity logarithm distribution in the output plane ($p = 0.5$) of the type I and type II systems for different aperture sizes with the incidence of a Gaussian beam. (a) $W_0 : a = 1 : 1$; (b) $W_0 : a = 1 : 1.5$; (c) $W_0 : a = 1 : 3$; (d) $W_0 : a = 1 : 5$.

$p = 0.5$ when a Gaussian beam passes the apertured fractional Fourier transforming systems for different aperture sizes is calculated by the analytical expression. Figure 4 shows relative intensity logarithm distribution in the output plane $p = 0.5$ of the type I and type II for different aperture sizes with the incidence of a Gaussian beam. It shows that the diffraction effect is weak in the case of $W_0 : a = 1 : 3$. The diffraction effect disappears in the case of $W_0 : a = 1 : 5$. In conclusion, the two optical setups for implementing FRFT are no longer equivalent in the presence of aperture effect.

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