Evolutions of perturbations with special frequencies in lossless optical fibers

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Expressing the perturbation optical field in terms of module and phase, using the linearized nonlinear Schrödinger equation governing the evolution of perturbations, we have deduced the analytical expressions of the modules, phases, and gain coefficients of the perturbations with zero or cut-off frequency, and studied the evolutions of the two perturbations travelling along lossless optical fibers in the negative dispersion regime. The results indicate that the phase of the perturbation with zero (or cut-off) frequency increases (or decreases) with the propagation distance monotonously and tends to its asymptotic value $n\pi + \pi/2$ (or $n\pi$) eventually. The evolution rates of the phases are closely related to the initial phase values. Although the asymptotic values of the field gain coefficients of the above mentioned two perturbations are equal to zero, and the increasing fashion of the modules is different from the familiar exponential type, it still suggests that the perturbations have a divergent nature when the propagation distance goes to infinity, indicating that the two kinds of perturbations can both lead to instability.

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Modulation instability in optics has attracted continuous interests[1−7]. Physically, its mechanics is identical to those of temporal[8] and spatial solitons[9,10]. In the temporal domain, the modulation instability can be interpreted as that, when co-propagating with the background optical field in the negative group velocity dispersion (GVD) region of the optical fibers, the perturbation field continuously obtains a gain from the background field and tends to divergent eventually as a result of the influence of the nonlinear effects. Previous investigations have made clear that the perturbation growth results from the interplay between the Kerr effects and the negative GVD. Utilizing the small signal approximation, the nonlinear Schrödinger equation has been linearized and the expression for the power gain coefficient of the perturbation with different angular frequencies has been derived. According to Refs. [11,12], in lossless optical fiber, only those perturbations with angular frequencies less than a critical value (or cut-off angular frequency) Ω_c can lead to modulation instability. And the value of the cut-off angular frequency Ω_c depends on the GVD coefficient β, Kerr coefficient γ, and the input power P_0 of the background optical field, which reads

$$\Omega_c = \sqrt{4\gamma P_0/|\beta|}. \tag{1}$$

The power gain coefficient for the perturbation with angular frequency Ω satisfies

$$g = |\beta\Omega| \sqrt{\Omega_c^2 - \Omega^2}. \tag{2}$$

According to Eq. (2), one concludes that, perturbations with angular frequencies $|\Omega|$ equal to 0 and Ω_c have zero power gain. In this case, one naturally concludes that it is unlikely that the modules of the two special perturbations will get larger and larger. Therefore, the two special perturbations should not be unstable.

Is it really true? Although it is difficult to verify this experimentally because loss always exists in optical fibers. The above mentioned conclusion cannot be made from the perturbation equations. In this paper, starting from the linearized nonlinear Schrödinger equation, evolutions of the two special perturbations are analyzed in detail. The results show that, although their modules do not grow exponentially as what we are familiar with, and although the equivalent gain coefficients vary with the propagation distances in the form of $1/z$ and tend to zero when the distance is long enough, the modules are divergent when the distance goes to infinity. Accordingly, the two perturbations should not be regarded as stable.

In the case of small signal approximation, the slowly varying module of the optical field including the perturbation can be assumed as[11,12]

$$A = (A_0 + \alpha)\exp\left(i\gamma A_0^2 z\right), \tag{3}$$

where A_0 and α are the modules of the input background optical field and the perturbation field, respectively. Substituting Eq. (3) into the nonlinear Schrödinger equation in lossless optical fibers, and in the case of small signal approximation, i.e., $|\alpha| \ll A_0$, adopting the first order approximation, one can obtain[11,12]

$$i\partial\alpha/\partial z = (\beta/2)\left(\partial^2\alpha\right)/\left(\partial T^2\right) - \gamma P_0(\alpha + \alpha^*), \tag{4}$$

where α^* is the complex conjugate of α. For the perturbation with angular frequency Ω, the perturbation α has once been written as $\{U_0\cos(kz-\Omega T)+iV_0\sin(kz-\Omega T)\}$ or $\{U_0\exp[i(kz - \Omega T)] + iV_0\exp[i(kz - \Omega T)]\}$, and the power gain coefficient has been derived for perturbation with different angular frequencies, as expressed in Eq. (2).

Mathematically, the above mentioned forms are usable in principle, but as the derivations are not so rigorous, some physical phenomena have been masked. Generally, as a complex number, the perturbation α can be expressed in the form of the real part and the imaginary

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part or two real numbers, i.e., the module and phase difference between the perturbation and the background optical field. Thus, the perturbation can be put down as

$$\alpha = \delta(z)\exp\{i[\Psi(z) - \Omega T]\}. \quad (5)$$

Substituting Eq. (5) into Eq. (4) and separating the real and imaginary part, one can obtain

$$\partial\varphi/\partial z = |\beta|\left\{\Omega_f^2[1+\cos(2\varphi)] - \Omega^2\right\}/2, \quad (6a)$$

$$\partial\delta/\partial z = |\beta|\,\Omega_f^2\delta\sin(2\varphi)/2, \quad (6b)$$

where

$$\varphi = \Psi(z) - \Omega T, \quad (7a)$$

$$\Omega_f = \Omega_c\big/\sqrt{2}. \quad (7b)$$

When deriving Eqs. (6a) and (6b), the fact of $\beta < 0$ has been taken into consideration. As a result, the two equations can only be used in negative dispersion optical fibers to describe the evolution of perturbation with angular frequency Ω. It can be seen from Eq. (6a) that, in the case of $|\Omega| < \Omega_c$, integration of ϕ leads to the inverse hyperbolic tangent function and one may conclude that the perturbation is unstable. The field gain coefficient g_E of the perturbation can be obtained directly from Eq. (6b) according to its definition

$$g_E = (1/\delta)(\partial\delta/\partial z). \quad (8)$$

In the special case, i.e., $|\Omega|$ equals to 0 or Ω_c, the solutions of Eq. (6a) will take different forms from those of $|\Omega| < \Omega_c$. Thus, the propagating stability of the two perturbations will take special forms. One will soon see from the following analysis that the two perturbations are divergent.

In case of $|\Omega| = \Omega_c$, from Eq. (6a) one can obtain that

$$\partial\varphi/\partial z = -|\beta|\Omega_f^2\sin^2\varphi. \quad (9)$$

Integration of Eq. (9) leads to

$$\mathrm{ctg}[\varphi(z)] = |\beta|\,\Omega_f^2 z + \mathrm{ctg}(\varphi_0), \quad (10)$$

where ϕ_0 is the initial value of $z = 0$. Inserting Eq. (10) into Eq. (6b), one can obtain

$$g_E = \left\{|\beta|\,\Omega_f^2\left[|\beta|\,\Omega_f^2 z + \mathrm{ctg}(\varphi_0)\right]\right\} \Big/ \left\{1 + \left[|\beta|\,\Omega_f^2 z + \mathrm{ctg}(\varphi_0)\right]^2\right\}. \quad (11)$$

It is easy to see that, g_E varies with the propagation distance z in the form of $1/z$, in other words, when the distance z tends to infinity, g_E of the perturbation with angular frequency Ω_c will tend to zero. However, this does not mean that the ratio of the module of the perturbation to its initial value is convergent.

Substituting Eq. (11) into Eq. (6b) and completing the integration, one can obtain

$$\delta(z) = \delta_0\sqrt{\left\{1 + \left[|\beta|\,\Omega_f^2 z + \mathrm{ctg}(\varphi_0)\right]^2\right\}\Big/\csc^2(\varphi_0)}. \quad (12)$$

It can be seen from Eq. (12) that, the module of the perturbation is divergent. In other words, the perturbation with angular frequency Ω_c should not be stable.

Similarly, it can also be proved that the phase and field gain coefficient of the perturbation with angular frequency 0 satisfy

$$\mathrm{tg}[\varphi(z)] = |\beta|\,\Omega_f^2 z + \mathrm{tg}(\varphi_0), \quad (13a)$$

$$g_E = \left\{|\beta|\,\Omega_f^2\left[|\beta|\,\Omega_f^2 z + \mathrm{tg}(\varphi_0)\right]\right\} \Big/ \left\{1 + \left[|\beta|\,\Omega_f^2 z + \mathrm{tg}(\varphi_0)\right]^2\right\}. \quad (13b)$$

The analytical expression of the perturbation module can still be obtained as

$$\delta(z) = \delta_0\sqrt{\left\{1 + \left[|\beta|\,\Omega_f^2 z + tg(\varphi_0)\right]^2\right\}\Big/\sec^2(\varphi_0)}. \quad (14)$$

From Eq. (14) it can be realized that the perturbation module is also divergent.

Equations (10)−(14) tell us that, at the initial stage of the evolution, the perturbation varies in a complicated fashion and the variation is closely related to the initial phase at $z = 0$, and the gain coefficient is not necessarily equal to zero. Only after propagating through a distance long enough, does the perturbation phase tend to an asymptotic value ϕ_∞ and the gain coefficient tend to zero.

Here, it is necessary to add that, the case of $\Omega = 0$ may correspond to the synthesis of two optical waves with different intensities, and the initial phase difference between them should be non-zero, otherwise, the perturbation analysis is not applicable. Similarly, one should not divide one optical wave into two parts with different intensities. When we use Eq. (3) to represent the optical field, the evolution of the main wave has been expressed as $A_0\exp(i\gamma A_0^2 z)$ and this expression is based on the assumption that the evolution of the main wave is not influenced by the perturbation. If one divides one optical wave into two parts and calls one of them the perturbation, the assumption that the evolution of the main wave is not influenced by the perturbation is not valid, because as a part of the main wave, the perturbation certainly influences its own evolution.

In Fig. 1, the phase evolutions of the perturbations, with angular frequency $|\Omega|$ equal to Ω_c (a) and 0 (b), have been plotted for different initial phases. The parameters used in calculations are $\beta = -0.02$ ps^2/m[11], $\Omega_f = 1\times10^{12}$ rad/s.

It can be seen that, the phases of the perturbations with cut-off angular frequency decrease monotonously with the propagation distance and tend to $n\pi$ ($n = 0, \pm1, \pm2, \cdots$) as shown in Fig. 1(a). When the initial phases fall into the regions of $-\pi < \phi_0 < 0$ and $0 < \phi_0 < \pi$, the asymptotic values ϕ_∞ of the perturbation phases are $-\pi$

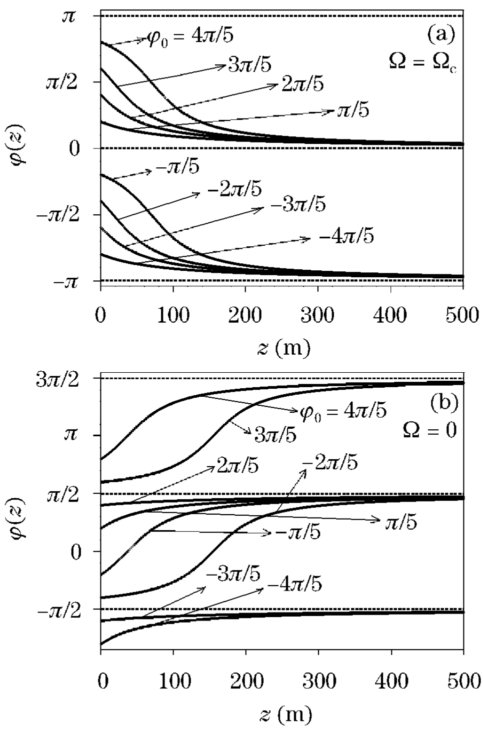

Fig. 1. Evolutions of perturbation phases with propagation distance for different initial phases.

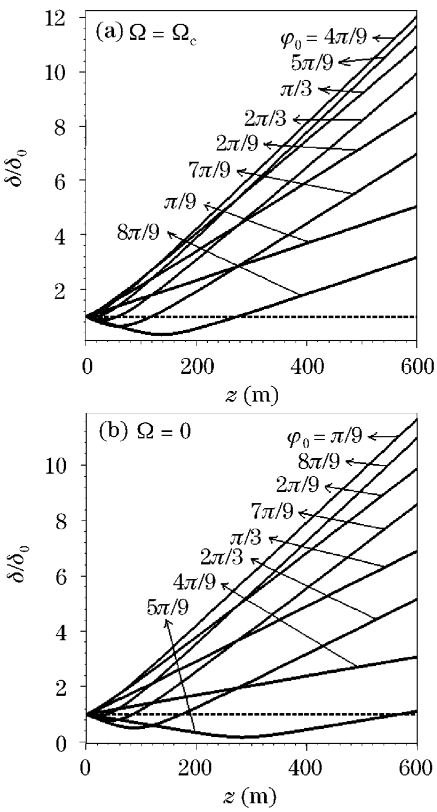

Fig. 2. Evolutions of perturbation modules with propagation distance for different initial phases.

and 0, respectively. Similarly, the phases of the perturbations with 0 angular frequency increase monotonously with the distance and tend to $n\pi + \pi/2$ ($n = 0, \pm 1, \pm 2, \cdots$) as shown in Fig. 1(b). When the initial phases fall into the regions of $-\pi/2 < \phi_0 < \pi/2$, $\pi/2 < \phi_0 < 3\pi/2$, and $-\pi < \phi_0 < -\pi/2$, the asymptotic values ϕ_∞ of the perturbation phases are $\pi/2$, $3\pi/2$, and $-\pi/2$, respectively.

It is not difficult to understand the above mentioned asymptotic behavior from the phase equations. For the perturbation with cut-off angular frequency, Eq. (10) indicates that, $\mathrm{ctg}[\phi(z)]$ is an increasing function of z. Thus, when the distance goes to infinity, the asymptotic value ϕ_∞ tends to $n\pi$ ($n = 0, \pm 1, \pm 2, \cdots$). In fact, Eq. (9) indicates that $\phi(z)$ itself is a decreasing function of z. Therefore, when the initial phases fall into the region $-\pi < \phi_0 < 0$ and $0 < \phi_0 < \pi$, the asymptotic values ϕ_∞ of the perturbation phases are $-\pi$ and 0, respectively. And the evolution tendency of the phase of the perturbation with zero angular frequency can also be analyzed in the similar way.

In Fig. 2, the evolutions of the normalized modules of the perturbations with angular frequency $|\Omega|$ equal to Ω_c (a) and 0 (b) have been plotted for different initial phases. The parameters of β and Ω_f used in calculations are the same as those in Fig. 1. It can be seen that, depending on the initial perturbation phases, the normalized modules of the two special perturbations evolve in two possible ways, increasing monotonously, and decreasing before increasing. In fact, from Eqs. (12) and (14), it can be realized that, within one period of the tangent and cotangent functions, if the initial phases satisfy $0 < \phi_0 < \pi/2$, the tangent and cotangent functions are both positive and then the perturbation modules will increase with the propagation distance monotonously; if the initial phases satisfy $\pi/2 < \phi_0 < \pi$, the two functions are both negative and then the perturbation modules will decrease before increasing. When the initial phases satisfy $\pi/2 < \phi_0 < \pi$, it is at the initial evolution stage $0 < z < |\mathrm{ctg}(\phi_0)|/(|\beta|\Omega_f^2)$ and $0 < z < |\mathrm{tg}(\phi_0)|/(|\beta|\Omega_f^2)$ for the perturbation with the cut-off angular frequency and zero angular frequency, respectively, that the normalized modules will decrease before increase monotonously.

If the perturbation field α is regarded as a phasor, the trajectories of the two perturbation phasors can be plotted for different initial phases in the polar coordinates. To make the variation tendency of the trajectories more intuitive, the natural logarithms of the normalized modules of the phasors are shown in Fig. 3, for $|\Omega| = \Omega_c$ (a) and $\Omega = 0$ (b). The initial phases (the angle between the rightward horizontal line and the line segment connecting the start point of the curve and the coordinate origin) corresponding to different trajectories used are: $\pi/8$, $\pi/4$, $3\pi/8$, $5.5\pi/8$, $6.5\pi/8$, $7.5\pi/8$, $-0.5\pi/8$, $-1.5\pi/8$, $-2.5\pi/8$, $-5\pi/8$, $-3\pi/4$, $-7\pi/8$ in Fig. 3(a) and $1.5\pi/8$, $2.5\pi/8$, $3.5\pi/8$, $-\pi/8$, $-\pi/4$, $-3\pi/8$, $-5.5\pi/8$, $-6.5\pi/8$, $-7.5\pi/8$, $-9\pi/8$, $-10\pi/8$, $-11\pi/8$ in Fig. 3(b). The parameters of β and Ω_f used in calculations are also the same as those in Fig. 1. It can be seen from Fig. 3(a) that, for the perturbations with cut-off angular frequencies, if the initial phases fall into the first and second quadrants, the phase angles will tend to $2n\pi$ ($n = 0, \pm 1, \pm 2, \cdots$) finally, otherwise, they will tend to $(2n + 1)\pi$ ($n = 0, \pm 1, \pm 2, \cdots$). Similarly, according to Fig. 3(b), for the perturbations with zero angular frequencies, the phase angles whose initial phases fall into the second and third

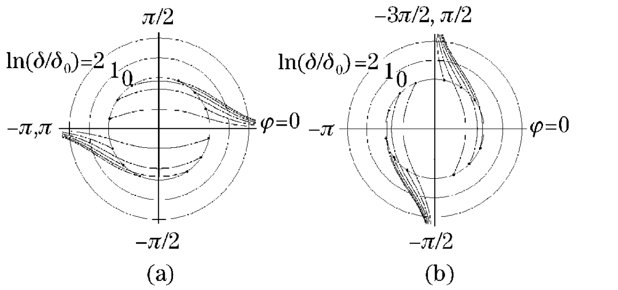

Fig. 3. The moving trajectories portrayed by the normalized perturbation phasors of natural logarithm for different initial phases.

quadrants will tend to $2n\pi - \pi/2$ ($n = 0, \pm1, \pm2, \cdots$) eventually, otherwise, they will tend to $2n\pi + \pi/2$ ($n = 0, \pm1, \pm2, \cdots$). It can still be seen from Fig. 3 that, whether the angular frequencies of the perturbations are 0 or Ω_c, if the initial phases fall into the first and third quadrants, the perturbation modules will increase monotonously, if the initial phases fall into the second and fourth quadrants, some parts of the trajectories will fall into the circle with the zero radius, which means that the perturbation modules will decrease before increasing. This increasing fashion of the perturbation modules is in accordance with that of Fig. 2 and the foregoing analytical discussions.

In terms of the module and phase to describe the optical perturbation field travelling in the negative dispersion regime of lossless optical fiber, using the linearized nonlinear Schrödinger equation governing the evolution of perturbations within the small signal regime, the analytical expressions for the modules, phases, and the field gain coefficients of the perturbations with zero and cut-off angular frequencies have been deduced. The evolutions of the modules and phases of the two special perturbations with the propagation distance are investigated. The results show that, although the field gain coefficients of the two special perturbations have zero asymptotic values, and the increasing fashion of the amplitudes is different from the familiar exponential type, the perturbations have a divergent nature when the propagation distance goes to infinity, indicating that the two perturbations can both lead to instability. In addition, when the distance gets long enough, the phases of the two special perturbations will tend to their asymptotic values. Depending on the different initial phases, some perturbation modules may increase with the distance monotonously, and some may decrease at first before increasing.

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