# Wavefront dislocations of Gaussian beams nesting optical vortices in a turbulent atmosphere

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A phase singularity of the light field created by interference of two Gaussian singular beams which propagate in a weak and near ground turbulent atmosphere is analyzed by the Rytov approximation and the short-term averaging method of the dislocation-position. We demonstrate that an edge or circular dislocation may be formed by both parallel and coaxial or noncoaxial collimated beams with different or equal beam-width interfere. The edge or circular short-term wavefront dislocations of super position field depend on the atmospheric turbulence strength, beam propagation distance, amplitude ratio, dislocation of nesting vortices, and beam-width or beam-width ratio of the individual beams.

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Optical fields may contain phase singularities or wavefront dislocations, that is to say, points or lines on the wavefront surface where the phase of the field is undetermined (singular) and its amplitude vanishes[1,2]. There are two types of phase dislocations. A screw dislocation or vortex is spiral phase ramps around a singularity, where the phase of the wave is undefined, thus its amplitude vanishes. The order of the screw dislocation multiplied by its sign is referred to the topological charge of the vortex. An edge dislocation is the $\pi$-shift in the wave phase located along a line in the transverse plane. The fascinating vorticity with important applications of the light field created by superposition of individual singular beams becomes an interesting question[1−10]. In the cases of superposition of two coaxial Gaussian (or Bessel-Gaussian or Laguerre-Gaussian) singular beams, the number of existing vortices and their net topological charge are found to depend on the beam widths and amplitudes during free-space propagation[3,4]. The dynamic inversion of the topological of superposition of Laguerre-Gaussian modes carrying different charges was observed under free-space propagation[5]. The superposition of vortices nested in Laguerre-Gaussian noncoaxial beams creates light patterns with much richer vortex content than that of the individual beams[6] and the vortical properties of the combined beam depend on propagation length as well as on amplitude ratio of individual beams during free-space propagation[7]. An optical vortex induces the splitting of an edge dislocation into vortices of both topological charges. Its position and number depend on which of the phase dislocations is shifted from the center of the host beam during free-space propagation[8]. The interference of two intersecting Laguerre-Gaussian singular beams creates vortices of light field under free-space propagation. The number and location of the vortices depend on the propagation length as well as on the topological charges of the individual beams, their intersection angle, and amplitude ratio[9]. Two or more beams nesting optical vortex interfere may form new composite optical vortices[10].

In this paper, we analyze the vorticity of the superposition field of two coherent Gaussian singular beams under near ground turbulent atmosphere propagation.

We consider an input field of Gaussian beam in the initial plane $z = 0$, which nests a single positive charge vortex that is located at $x = 0$, $y = 0$. The field can be described as

$$U_0(\mathbf{r},0) = U_0(x+iy)\exp\left(-\frac{r^2}{w_0^2} - i\frac{kr^2}{2R_0}\right), \qquad (1)$$

where $k = 2\pi/\lambda$ is wave number, $\lambda$ is wavelength, $w_0$ is waist of beam, $R_0$ is the radius of the wave front curvature in its waist, $U_0$ is the amplitude parameter, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors, and $x$ and $y$ are the transverse coordinate.

By $ABCD$ law of beam propagation through an optical system, an amplitude of the singular light beam after propagating a distance $L$ in free-space, can be written as[8,11]

$$U_0(\mathbf{r},L) = \frac{w_0 U_0 r}{w(L)}\exp\left[-\frac{r^2}{w^2(L)}\right]\exp\left[i\Phi(\mathbf{r},L)\right], \qquad (2)$$

where $w(L) = w_0\left[(1-L/R_0)^2 + \left[\lambda L/\left(\pi w_0^2\right)\right]^2\right]^{1/2}$, $R = \frac{R_0\left[(1-L/R_0)^2+\left[\lambda L/\left(\pi w_0^2\right)\right]^2\right]}{\left((1-L/R_0)-\left[\lambda L/\left(\pi w_0^2\right)\right]^2\right)}$, $\Phi(\mathbf{r},L) = kL - \frac{kr^2}{2R} + \arctan\frac{L}{z_R} + \arctan\frac{y}{x}$, $z_R = kw_0^2/2$ is the Rayleigh rang, $\arctan(L/z_R)$ is an additional phase, which is called Gouy phase shift. The parameter $1 - L/R_0$ is called the curvature parameter. $1 - L/R_0 = 1$, $1 - L/R_0 > 1$, and $1 - L/R_0 < 1$, correspond to the collimated, divergent, and convergent beam forms, respectively. When the vortex is located at $x = x_0$ and $y = y_0$, the vortex field after propagating a distance $L$ can be given by[7,9]

$$\tilde{U}_0(\tilde{\mathbf{r}},L) = \frac{w_0 U_0 \tilde{r}}{w(L)}\exp\left[-\frac{\tilde{r}^2}{w^2(L)}\right]\exp\left[i\Phi(\tilde{\mathbf{r}},L)\right], \qquad (3)$$

where $\tilde{\mathbf{r}} = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j}$.

If a random medium exists between input and output planes, the optical field at $z = L$ under the Rytov approximation is[12]

$$U_\varphi(\tilde{\mathbf{r}},L) = \tilde{U}_0(\tilde{\mathbf{r}},L)\exp\psi(\tilde{\mathbf{r}},L), \qquad (4)$$

where $\tilde{U}_0(\tilde{\mathbf{r}}, L)$ denotes the vortex optical field (in Eq. (3)) and $\psi(\tilde{\mathbf{r}}, L)$ is the complex phase perturbation of the field due to random inhomogeneities along the propagation path. It is customary in the Rytov approximation to write[12]

$$\psi(\tilde{\mathbf{r}}, L) = \chi(\tilde{\mathbf{r}}, L) + iS(\tilde{\mathbf{r}}, L), \tag{5}$$

where $\chi$ and $S$ denote the log amplitude fluctuation and the phase fluctuation, respectively, of the complex phase perturbation.

In what follows, we analyze the interference of the parallel and noncoaxial Gaussian beams with nested single-charged vortices. Without loss of generality, two positively single-charged vortices and beam centers are assumed to be located along the $y$ axis, respectively, and separated by distance $2y_0$ from each other at $z = 0$. The amplitude of combined Gaussian beam can be written as

$$\begin{aligned} U_\varphi(\tilde{\mathbf{r}}, L) &= U_{\varphi 1}(\tilde{\mathbf{r}}_1, L) + U_{\varphi 2}(\tilde{\mathbf{r}}_2, L) \\ &= \frac{w_{01}U_{01}\tilde{r}_1}{w_1^2(L)} \exp\left[i\tilde{\Phi}_1(\tilde{\mathbf{r}}_1, L) - \frac{\tilde{r}_1^2}{w_1^2(L)} + \chi_1\right] \\ &+ \frac{w_{02}U_{02}\tilde{r}_2}{w_1^2(L)} \exp\left[i\tilde{\Phi}_2(\tilde{\mathbf{r}}_2, L) - \frac{\tilde{r}_2^2}{w_2^2(L)} + \chi_2\right], \end{aligned} \tag{6}$$

where $\tilde{\Phi}_i(\tilde{\mathbf{r}}_i, L) = -kL - \frac{k\tilde{r}_i^2}{2R_i} + S_i(\tilde{\mathbf{r}}_i, L) + \arctan\frac{L}{z_{Ri}} + \arctan\frac{y \pm y_0}{x} (i = 1, 2)$, here the minus sign "$-$" for $i = 1$ and the plus sign "$+$" for $i = 2$, and $\tilde{r}_i = \sqrt{x^2 + (y \pm y_0)^2}$.

We know that the wavefront dislocations in the combined beam are located at the phase's relationship $\tilde{\Phi}_1 = \tilde{\Phi}_2 \pm \pi$ and zero-amplitude of the field $U_\varphi(\tilde{\mathbf{r}}, L) = 0$[4]. The dislocation position (zero amplitude of the field) is determined by

$$\begin{aligned} 0 = &\frac{w_{01}U_{01}\tilde{r}_1}{w_1^2(L)} \exp\left[-\frac{\tilde{r}_1^2}{w_1^2(L)} + \chi_1 - \chi_2\right] \\ &- \frac{w_{02}U_{02}\tilde{r}_2}{w_2^2(L)} \exp\left[-\frac{\tilde{r}_2^2}{w_2^2(L)}\right]. \end{aligned} \tag{7}$$

Since $\chi_i$ $(i = 1, 2)$ in Eq. (7) are random variables and the signal fields $U_\varphi$ are measured during finite time, we must make use of the ensemble-average of atmosphere for calculating the position of the wavefront dislocations (PWFD). The PWFD obtained by this method is referred to ensemble-average PWFD. The theoretical and experimental researches for the frequency spectrum of the phase fluctuations and the frequency spectrum of the intensity fluctuations in Refs. [13] and [14] show that the frequency of the intensity fluctuations is more high than that of the phase fluctuations, that is to say, comparing with the intensity fluctuations, the phase fluctuations are slower. Let $\tau_{\mathrm{I}}$ be the period of the intensity fluctuations and $\tau_{\mathrm{p}}$ be the period of the phase fluctuations, when the signal fields are measured during $\tau_{\mathrm{I}} \ll \tau_{\mathrm{s}} < \tau_{\mathrm{p}}$, we can take the phase of wave field as constant phase, approximately. By above approximation, the PWFD measured during $\tau_{\mathrm{s}}$ is an ensemble-average short-exposure PWFD, for simplicity, we call it short-term wavefront dislocation.

In the regime where Rytov approximation is suitable and $\chi$ is a Gaussian random variable, we can use the well-known theorem[12]

$$\langle \exp\chi \rangle = \exp\left[\langle \chi \rangle + \frac{1}{2}\left\langle (\chi - \langle \chi \rangle)^2 \right\rangle\right] \tag{8}$$

to evaluate the averages in Eq. (7).

Because the wander of the beam propagation in weak turbulent atmosphere is small[15] and the phase fluctuation as well as the beam wander are slow, the average of the position coordination $\langle \tilde{r}_i \rangle_{\mathrm{s}}$ during $\tau_{\mathrm{s}}$ is much larger than the position fluctuation (or position deviation) $\Delta r$, i.e., $\langle \tilde{r}_i \rangle_{\mathrm{s}} \gg \Delta r = \tilde{r}_i - \langle \tilde{r}_i \rangle_{\mathrm{s}}$ and $\tilde{r}_i^2 \approx \langle \tilde{r}_i \rangle_{\mathrm{s}}^2$, here the footnote "s" represents short-term average. Then, taking the ensemble-average of the short-exposure for both sides of Eq. (7) and substituting Eq. (8) into it, we have the equation for the ensemble-average short-exposure position of the wavefront dislocations

$$\begin{aligned} 0 = &\frac{w_{01}U_{01}\tilde{r}_1}{w_1^2(L)} \exp\left[-\frac{\tilde{r}_1^2}{w_1^2(L)} - \frac{1}{2}D_\chi\right] \\ &- \frac{w_{02}U_{02}\tilde{r}_2}{w_2^2(L)} \exp\left[-\frac{\tilde{r}_2^2}{w_2^2(L)}\right], \end{aligned} \tag{9}$$

where $D_\chi = \left\langle (\chi_1 - \chi_2)^2 \right\rangle$ is the structure function of the log-amplitude fluctuation $s$. For homogeneous and isotropic random field, $D_\chi \approx 0.25k^{7/6}C_n^2 L^{11/6}$, $C_n^2$ is the structure constant.

Let us first consider the propagation of the parallel and noncoaxial beams. Two beams are located along the $y$ axis by a distance $\pm y_0$ from the origin, respectively. Consider two beams with equal beam-width, that is $w_{01} = w_{02}$, $w_1^2(L) = w_2^2(L) = w^2(L)$. In addition, let the nested vortices be located at $x = 0$, and $y = \pm y_0$, respectively, and assume the distance $y_0$ between the centers of beams as well as vortices $z$ is a small value. Making use of these approximations and Eq. (9), in Cartesian coordinates, we obtain

$$\begin{aligned} 0 \cong &U_{10} \exp\left[-\frac{x^2 + (y - y_0)^2}{w^2(L)} - \frac{1}{2}D_\chi\right] \\ &- U_{20} \exp\left[-\frac{x^2 + (y + y_0)^2}{w^2(L)}\right]. \end{aligned} \tag{10}$$

From Eq. (10), we have the short-term average position of the wavefront dislocation line

$$y_{\mathrm{s}} = w^2(L)\left(\ln U_{20}/U_{10} + 0.125k^{7/6}C_n^2 L^{11/6}\right)/4y_0. \tag{11}$$

Equation (11) demonstrates that an edge dislocation of the combined beam can be created by the interference of two equal beam-width, noncoaxial and parallel Gaussian beams nesting a positively single-charged vortices, the location of the edge dislocation depends on the propagation length, strength of the atmospheric turbulence, amplitude ratio, dislocation of nested vortices, and beam-width of the individual beams. Equation (11) is an extension of the investigative result in Ref. [10] for singularity of interference filed of two singular beams propagation in free-space.

Let us next consider the propagation of the parallel and coaxial collimated beams with different beam-width. The positions of vortices nested by Gaussian beams are located at $x = 0$ and $y = 0$, the centers of host beams are also located at $x = 0$, $y = 0$, and $\tilde{r}_1 = \tilde{r}_2 = r$. From Eq. (9), we obtain

$$0 = \frac{w_{01}U_{01}}{w_1^2(L)} \exp\left[-\frac{r^2}{w_1^2(L)} - \frac{1}{2}D_\chi\right] - \frac{w_{02}U_{02}}{w_2^2(L)} \exp\left[-\frac{r^2}{w_2^2(L)}\right], \quad (12)$$

In Eq. (12), the approximation $\langle r \rangle \gg \Delta r = r - \langle r \rangle$, $r^2 = r_{\rm s}^2$ was used. The short-term average-radius of the zero-amplitude circular called circular dislocation[4] is given as

$$r_{\rm s} = \left\{ \frac{w_1^2(L) w_2^2(L)}{w_1^2(L) - w_2^2(L)} \times \left[ \ln \frac{w_{02}U_{02}w_1(L)}{w_{01}U_{01}w_2(L)} + 0.125 k^{7/6} C_n^2 L^{11/6} \right] \right\}^{1/2}. \quad (13)$$

The result in Eq. (13) demonstrates that the short-term wavefront circular dislocation arising from the interfere of two Gaussian beams nesting vortex, with the vortices located at $x = 0$ and $y = 0$, is the same as the short-term wavefront circular dislocation arising from the interfere of two smooth Gaussian beams. The short-term wavefront circular dislocation depends on the propagation length, strength of the atmospheric turbulence, amplitude ratio, dislocation of vortices, and beam-width ratio of the individual beams. Equation (13) also shows that the short-term wavefront circular dislocation is proportion to strength of the atmospheric turbulence $C_n^2$ if the propagation length, the amplitude ratio, the beam-width ratio, and the wavelength of beams were given. That is to say, if the structure function of the log-amplitude fluctuation $D_\chi \approx 0.25 k^{7/6} C_n^2 L^{11/6}$ is very small in very weak fluctuation region, the effects of the atmospheric turbulence on the short-term wavefront circular dislocation arising from the interference of singular beams are also very small.

In conclusions, our investigation of the superposition of two singular beams has revealed that the transverse position of the resulting composite vortex can be controlled by selecting a control parameter such as the amplitude ratio, or the beam-width ratio, or distance between the composing beams etc.. We found that the interference of two equal beam-width, noncoaxial and parallel Gaussian beams nesting a positively single-charged vortices may form an edge dislocation, and the location of the edge dislocation depends on the propagation length, strength of the atmospheric turbulence, amplitude ratio, dislocation of vortices, and beam-width of the individual beams. The superposition of the parallel and coaxial singular collimated beams with different beam-width may produce the circular dislocation that the center is at propagation axis and the short-term wavefront circular dislocation depends on the propagation length, strength of the atmospheric turbulence, amplitude ratio, dislocation of vortices, and beam-width ratio of the individual beams. The analytical result in Eq. (13) demonstrates that the index-of-refraction structure constant $C_n^2$ can be measured by the measurement of the short-term wavefront circular dislocation $r_{\rm s}$, when the amplitude ratio, the beam-width ratio, and the propagation length were selected.

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