Non-local implementation of single-qubit rotation operation

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We present two optimal schemes for non-local implementing a single-qubit rotation operation via a maximally entangled quantum channel. We report on the quantitative relations between the quantum action, entangled and classical communication resources required in the implementation. We also put forward two schemes for conclusive implementing the non-local quantum single-qubit rotation via a partially entangled quantum channel. Both these methods can appropriately be referred to as qubit-assisted processes.
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The quantum non-locality, i.e. there are non-local correlations among quantum systems, plays a central role in quantum computation and quantum information. In the past, most research on quantum non-locality has been devoted to the issue of non-locality of quantum states[1−7]. However, an equally important issue is that of non-locality of quantum operations. This situation arises, for example, in the context of distributed quantum computer[8], where two or more spatially separated computation units are available to solve a computation problem. Another examples where non-local quantum operations are required are quantum network communication[9] and the production of multi-particle entangled states[10]. For the alluring potential application, some fundamental research on non-local quantum operation has been presented[11−15].

It is proved that any unitary operation on N qubits may be implemented exactly by composing single-qubit rotations and control-Not gates[8]. This implies that the single-qubit rotation is an especially important essential operation, and the resource requirement for implementing the operation is one of the limiting factors in the construction of general unitary transformation in quantum computational and informational networks. In this letter, we will investigate the non-local implementation of a quantum single-qubit rotation. We will restrict our attention to the rotations about $\hat{Z}$ axis

$$U_\theta \equiv \exp(-i\theta\sigma_z) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}, \tag{1}$$

for an arbitrary angle $\theta \in (0, 2\pi]$. Note that $U_\pi = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -iZ$; $U_{\pi/4} = e^{-i\pi/8}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8}T$; and $U_{\pi/4}^2 = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{i\pi/4}S$. The Pauli operator Z, $\pi/8$ gate T and phase gate S are some of the most important essential gates.

Consider Alice and Bob are localized in different space domains, who are able to perform local operations. They also have addition resources, namely they share entangled state, and they are able to communicate classically. Alice possesses a device which is able to perform U_θ, whereas Bob has a qubit in the unknown state

$$|\psi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B. \tag{2}$$

Their goal is that Bob ends up with the processed state

$$U_\theta|\psi\rangle_B = \alpha e^{-i\frac{\theta}{2}}|0\rangle_B + \beta e^{i\frac{\theta}{2}}|1\rangle_B. \tag{3}$$

First of all, it is important to note that any unitary operation can be implemented non-locally given enough shared entanglement and classical communication. Consider the case of single-qubit rotation, the unitary transformation U_θ on Bob's qubit $|\psi\rangle_B$ can be accomplished by Bob teleporting his qubit to Alice, Alice performing U_θ locally and finally Alice teleporting Bob's qubit back to Bob. The resources needed for the two teleportation actions are: "one e-bit puls two classical bits (c-bit) transmitted from Bob to Alice for the Bob to Alice teleportation" plus "one e-bit plus two c-bits from Alice to Bob for the Alice to Bob teleportation". The "double teleportation"[16] procedure shown above is sufficient to implement any quantum evolution. The question is, however, whether so many resources are actually needed. In this letter, we firstly present two perfect schemes by which the single-qubit rotation can be implemented non-locally using an Einstein-Podolsky-Rosen (EPR) pair as quantum channel. The consumption of overall physical resources is compared for the two schemes. Then, we propose other two schemes of non-local implementation of single-qubit rotation via a partially entangled quantum channel, and the probability of successful implementation is given.

Suppose Alice and Bob previously share a EPR pair

$$|\phi\rangle_{ba} = 1/\sqrt{2}(|00\rangle_{ba} + |11\rangle_{ba}). \tag{4}$$

The particles a and b belong to Alice and Bob, respectively. The initial state of the whole system is

$$|\psi\rangle_{Bba} = 1/\sqrt{2}(\alpha|0\rangle_B + \beta|1\rangle_B)(|00\rangle_{ba} + |11\rangle_{ba}). \tag{5}$$

Implementation scheme 1.1 is defined as two-way classical communication scheme.

Step 1: A local CNOT operation is applied between Bob's qubits with the B acting as the control qubit following by a projective measurement of the target qubit b in the computation basis. Qubit b is subsequently discarded. This sequence consumes one c-bit from Bob to Alice. If the result is $|0\rangle_b$, Alice does nothing; and if the result is $|1\rangle_b$, Alice performs the X (here and below, we will write X and Z instead of σ_x and σ_z.) operation on qubit a. After these operations, the state given by Eq. (5) will become

$$1/\sqrt{2}(\alpha|00\rangle_{Ba} + \beta|11\rangle_{Ba}). \tag{6}$$

Step 2: Alice applies the operation U_θ onto her qubit a followed by a Hadamard transformation. The global state of the distributed system after this action can be written as

$$\frac{1}{2}\alpha e^{-i\frac{\theta}{2}}|0\rangle_B(|0\rangle_a+|1\rangle_a)+\frac{1}{2}\beta e^{i\frac{\theta}{2}}|1\rangle_B(|0\rangle_a-|1\rangle_a)$$

$$=\frac{1}{2}U_\theta(\alpha|0\rangle+\beta|1\rangle)_B|0\rangle_a+\frac{1}{2}Z_BU_\theta(\alpha|0\rangle+\beta|1\rangle)_B|1\rangle_a. \quad (7)$$

A projective measurement in computation basis on Alice's side yields a collapsed state on Bob's side which is either the state shown in Eq. (3) whenever the measurement outcome is $|0\rangle_a$, or a state that can be locally transformed into the state (3). If the measurement outcome is $|1\rangle_a$, all Bob has to do is applying the correcting operation Z on qubit B. Bob needs to know Alice's measurement outcome therefore a further c-bit is consumed in the second part of the scheme.

Synthesizing all conditions (four kinds), we obtain the probability of successful implementation is $P=(1/2)^2\times 4=1$. We have thus shown how to non-local implement a single-qubit rotation and seen that it consumes one e-bit and one c-bit in each direction.

Implementation scheme 1.2 is defined as one-way classical communication scheme.

The initial state of the whole system is still in state (5).

Step 1: Bob performs a control-Rotation operation R_{Bb} on his qubits B and b (with the control being qubit B)

$$R_{Bb}=|00\rangle\langle 00|+|01\rangle\langle 01|-|10\rangle\langle 11|+|11\rangle\langle 10|,$$

then Eq. (5) will be transformed into

$$1/\sqrt{2}(\alpha|00\rangle_{Bb}+\beta|11\rangle_{Bb})|0\rangle_a$$
$$+1/\sqrt{2}(\alpha|01\rangle_{Bb}-\beta|10\rangle_{Bb})|1\rangle_a. \quad (8)$$

Step 2: Alice performs the rotation operation U_θ on his qubit a, and then a Hadamard transform, after that, Expression (8) will be

$$\frac{1}{2}(\alpha|00\rangle_{Bb}+\beta|11\rangle_{Bb})e^{-i\frac{\theta}{2}}(|0\rangle_a+|1\rangle_a)$$
$$+\frac{1}{2}(\alpha|01\rangle_{Bb}-\beta|10\rangle_{Bb})e^{i\frac{\theta}{2}}(|0\rangle_a-|1\rangle_a). \quad (9)$$

Then, Alice performs a computation basis measurement on his qubit a and qubit a is subsequently discarded. If the result is $|0\rangle_a$, particles B and b are located in the state

$$\frac{1}{2}(\alpha|0\rangle_B e^{-i\frac{\theta}{2}}-\beta|1\rangle_B e^{i\frac{\theta}{2}})|0\rangle_b$$
$$+\frac{1}{2}(\alpha|0\rangle_B e^{i\frac{\theta}{2}}+\beta|1\rangle_B e^{-i\frac{\theta}{2}})|1\rangle_b. \quad (10a)$$

Otherwise, they will be located in the state

$$\frac{1}{2}(\alpha|0\rangle_B e^{-i\frac{\theta}{2}}+\beta|1\rangle_B e^{i\frac{\theta}{2}})|0\rangle_b$$
$$-\frac{1}{2}(\alpha|0\rangle_B e^{i\frac{\theta}{2}}-\beta|1\rangle_B e^{-i\frac{\theta}{2}})|1\rangle_b. \quad (10b)$$

Alice informs Bob of the measurement result, which is one c-bit of information.

Step 3: Suppose Alice's measurement result is $|0\rangle_a$. From this Bob concludes that the state of particles B and b is Expression (10a). Then Bob performs a computation basis measurement on his qubit b. If the measurement outcome is $|1\rangle_b$, Expression (10a) will be

$$\frac{1}{2}(\alpha|0\rangle_B e^{i\frac{\theta}{2}}+\beta|1\rangle_B e^{-i\frac{\theta}{2}})=\frac{1}{2}U_\theta^{+}(\alpha|0\rangle_B+\beta|1\rangle_B), \quad (11)$$

the implementation fails. If the measurement outcome is $|0\rangle_b$, Bob performs a Pauli operator Z on qubit B and Expression (10a) will be

$$\frac{1}{2}(\alpha|0\rangle_B e^{-i\frac{\theta}{2}}+\beta|1\rangle_B e^{i\frac{\theta}{2}})=\frac{1}{2}U_\theta(\alpha|0\rangle_B+\beta|1\rangle_B). \quad (12)$$

This completes the non-local single-qubit rotation with probability $P=(1/2)^2\times 2=1/2$ (2 kinds all). Remarkable enough, in this scheme only one-way communication (one c-bit) is required (in addition to entanglement (one e-bit)) in order to remotely "rotate" Bob's qubit by Alice, as opposed to the two-way classical communication of the scheme 1.1. The price to be paid, however, is that the scheme 1.2 only succeeds with probability 1/2.

In real situations Alice and Bob may not have shared maximally entangled state but partially entangled state (due to some imperfection at the source). This means that the quantum channel for implementing will be imperfect. Usually if one follows the above schemes, one will not be able to complete the implementation process with unit fidelity. Of course, if one has several partially entangled pairs one can first perform entanglement concentration[17] and then recover fewer perfect maximally entangled pairs, and then use one of them to implement the operation using the above schemes. If Alice and Bob have only one pair, there are two strategies to non-local implement the operation U_θ via partially entangled channel: 1) purifying the partially entangled channel to a maximally entangled channel before using it by entanglement concentration then implementing the operation; 2) implementing the operation through the partially entangled channel directly then rectifying the distorted operation. In the following, we will use the second strategy to investigate how the operation U_θ can be implemented non-locally with unit fidelity albeit with reduced probability by using partially entangled state as quantum channel. We propose two explicit schemes for this purpose. We will see that both these methods can appropriately be referred to as qubit-assisted processes, since in both proposes either Alice or Bob is required to prepare a qubit in some specified state to implement the respective protocol.

Suppose Alice and Bob share a partially entangled state

$$|\phi\rangle_{ba}=a|00\rangle_{ba}+b|11\rangle_{ba}\ (|a|^2+|b|^2=1,|a|\geq|b|). \quad (13)$$

The initial state of the whole system is

$$|\psi\rangle_{Bba}=(\alpha|0\rangle_B+\beta|1\rangle_B)(a|00\rangle_{ba}+b|11\rangle_{ba}). \quad (14)$$

Implementation scheme 2.1 is that Alice introduces an ancilla qubit.

The goal of this proposal is to modify the measurement part of Alice so that after Alice carries out her specific measurement and communicates her result, the state of Bob will be given by state (3).

Step 1: The first step of the scheme is exactly the same as the one shown in scheme 1.1. After this, Eq. (14) will become

$$a\alpha|00\rangle_{Ba} + b\beta|11\rangle_{Ba}$$

$$\text{(if Bob's measurement result is } |0\rangle_b), \quad (15a)$$

or

$$b\alpha|00\rangle_{Ba} + a\beta|11\rangle_{Ba}$$

$$\text{(if Bob's measurement result is } |1\rangle_b). \quad (15b)$$

Suppose Bob's measurement result is $|0\rangle_b$.

Step 2: Alice applies the operation U_θ onto her qubit a, and the state of the distributed system after this action can be written as

$$a\alpha e^{-i\frac{\theta}{2}}|0\rangle_B|0\rangle_a + b\beta e^{i\frac{\theta}{2}}|1\rangle_B|1\rangle_a. \quad (16)$$

Step 3: Alice now prepares an ancilla qubit a' in the state

$$|\chi\rangle_{a'} = |0\rangle_{a'} + |1\rangle_{a'}. \quad (17)$$

The combined state of the three qubits is given by

$$|\psi\rangle_{Baa'}$$
$$= (a\alpha e^{-i\frac{\theta}{2}}|0\rangle_B|0\rangle_a + b\beta e^{i\frac{\theta}{2}}|1\rangle_B|1\rangle_a)(|0\rangle_{a'} + |1\rangle_{a'})$$
$$= \frac{1}{2}\{[U_\theta(\alpha|0\rangle + \beta|1\rangle)_B](a|00\rangle + b|11\rangle)_{aa'}$$
$$+[Z_B U_\theta(\alpha|0\rangle + \beta|1\rangle)_B](a|00\rangle - b|11\rangle)_{aa'}$$
$$+[U_\theta(\alpha|0\rangle + \beta|1\rangle)_B](a|01\rangle + b|10\rangle)_{aa'}$$
$$+[Z_B U_\theta(\alpha|0\rangle + \beta|1\rangle)_B](a|01\rangle - b|10\rangle)_{aa'}\}. \quad (18)$$

Now measurement on Alice's side takes place in two steps. The first measurement projects the state onto either of the subspaces spanned by $\{|00\rangle, |11\rangle\}$ or $\{|01\rangle, |10\rangle\}$. Thus this measurement has two possible outcomes that occur with equal probability. Suppose the result is in the subspace spanned by $\{|00\rangle, |11\rangle\}$. Alice now performs an optimal positive operator value measure[8] (POVM), which provides the most general physically realizable measurement in quantum mechanics. The generalized measurement can distinguish conclusively between the two nonorthogonal states $(a|00\rangle + b|11\rangle)$ and $(a|00\rangle - b|11\rangle)$. Having the measurement result on qubit a and a', Bob can transform the state of his qubit into the state (3).

In step 1, if Bob's measurement result is $|1\rangle_b$, it can be discussed in the same way.

Because there is always a possibility of an inconclusive result in the state discrimination procedure, the total probability of successful implementation is $2|b|^2$. Thus, we can say that using $E(|\varphi\rangle_{ba}) = (-a^2\log_2 a^2 - b^2\log_2 b^2)$ amount of entanglement, one c-bit from Bob to Alice and two c-bits from Alice to Bob, Alice can remotely "rotates" Bob's qubit with unit fidelity and probability $2|b|^2$. Note that for $b = 1/\sqrt{2}$, which corresponds to a maximally entangled channel, the proposal is always successful with certainty as there need not be any inconclusive result since in this case one discriminates between two orthogonal states.

The previous proposal suggested changes in the type of measurement by Alice. In the following proposal we keep the measurement part of Alice intact but modify Bob's local implementation.

Implementation scheme 2.2 is that Bob introduces an ancilla qubit.

Step 1 – Step 2: The first two steps of the scheme are exactly the same as the ones shown in scheme 1.1. After this, Eq. (14) will become

$$\frac{1}{\sqrt{2}}U_\theta(a\alpha|0\rangle_B + b\beta|1\rangle_B)$$

$$\text{(in step 1, if Bob's measurement result is } |0\rangle_b), \quad (19a)$$

or

$$\frac{1}{\sqrt{2}}U_\theta(b\alpha|0\rangle_B + a\beta|1\rangle_B)$$

$$\text{(in step 1, if Bob's measurement result is } |1\rangle_b). \quad (19b)$$

But Bob finds they yet cannot evolve the state (2) to the state (3) which they need, because the states (19) include the parameters of the imperfect quantum channel, a and b. In order to distill the state (3) from the states (19), Bob must give some evolution to rectify the distorted state.

Suppose Bob's measurement result is $|0\rangle_b$ in the step 1.

Step 3: Bob prepares an ancilla qubit b' in a state $|0\rangle_{b'}$. Thus the combined state of the two qubits that Bob holds is now given by

$$|\psi\rangle_{Bb'} = \frac{1}{\sqrt{2}}U_\theta(a\alpha|00\rangle + b\beta|10\rangle)_{Bb'}. \quad (20)$$

Bob now performs a CNOT operation on his two qubits states (with the control being qubit B), thus transforming it into the state

$$|\psi'\rangle_{Bb'} = \frac{1}{2\sqrt{2}}\{(a|0\rangle + b|1\rangle)_B[U_\theta(\alpha|0\rangle + \beta|1\rangle)_{b'}]$$
$$+(a|0\rangle - b\beta|1\rangle)_B[Z_{b'}U_\theta(\alpha|0\rangle + \beta|1\rangle)_{b'}]\}. \quad (21)$$

Form Eq. (21) it is clear that an optimal POVM , which can conclusively distinguish between the two nonorthogonal states $(a|0\rangle + b|1\rangle)$ and $(a|0\rangle - b|1\rangle)$, will give the desired result. Note that the POVM can be carried out on any one of the two qubits that Bob holds.

In step 1, if Bob's measurement result is $|1\rangle_b$, it can be discussed in the same way. Synthesizing all cases, we obtain the total probability of obtaining a conclusive result is $2|b|^2$.

It may be noted that, in the sense of successful implementation probability, implementing the non-local single-qubit rotation operation directly through the partially entangled channel is always better than the strategy based on purification of the channel firstly. For a single copy, the best purification is the method shown in Ref. [17], which has optimal efficiency $2|b|^2$. If we restrict performance of purification to the use of linear devices, the efficiency will be less than $2|b|^2$. That is, if we purify the partially entangled quantum channel before using it to implement the rotation operation based on linear devices, the probability of successful implementation is always less than $2|b|^2$. On the other hand, although the probabilistic operations may be not so useful in the context of quantum computation, as it may change the complexity class of the problem and may thus destroy the (exponential) speed up of the quantum algorithm in question. However, probabilistic gates are useful for processes such as entanglement distillation[8], which itself is already a probabilistic process. For example, this may help in the implementation of quantum repeaters[18,19] using photons only (i.e., for quantum communication over arbitrary distance). Due to the fact that photons are ideal candidates for quantum information processing (due to their fast propagation), it is highly desirable to manipulate them directly rather than mapping their states on the states of another physical system, e.g. of an ion or an atom, and vice versa.

In summary, the schemes for non-local implementing a single-qubit rotation operation by different quantum channels have been presented. One quantum channel is that Alice and Bob share the EPR pair. By performing the local operations and projective measurements, Alice can remotely "rotate" Bob's qubit. Comparing the schemes 1.1 and scheme 1.2, the first one gains an advantage over the other in the successful implementation probability $P = 1$, but it is a two-way communication scheme. The second one reduces the number of c-bit to one, which consumes less resource at the expense of the less success probability $P = 1/2$. Making use of a partially entangled quantum channel, the non-local implementation of a single-qubit rotation operation can be realized with unit fidelity albeit with reduced probability. In both scheme 2.1 and scheme 2.2, either Alice or Bob is required to prepare a qubit in some specified state to implement the respective protocols. The probability of success is equal to the twice modulus square of the smaller coefficient of the entangled two-particle state.

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References

1. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70,** 1895 (1993).
2. D. Bouwmeester, J. W. Pan, K. Mattle, M. Eibl, H.Weinfuter, and A. Zeilinger, Nature **390,** 575 (1997).
3. J. M. Liu, J. Li, and G. C. Guo, Chin. Opt. Lett. **1,** 114 (2003).
4. H. Lu, Chin. Phys. Lett. **18,** 1004 (2001).
5. H. Lu and G. C. Guo, Phys. Lett. A **276,** 209 (2000).
6. X. Lin and H. C. Li, Acta Opt. Sin. (in Chinese) **23,** 137 (2003).
7. L. B. Chen, Chin. Phys. **11,** 999 (2002).
8. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
9. J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Phys. Rev. Lett. **78,** 3221 (1997).
10. S. Bose, V. Vedral, and P. L Knight, Phys. Rev. A **57,** 822 (1998).
11. D. Gottesman and I. L. Chuang, Nature **402,** 390 (1999).
12. J. Eisert, K. Jacobs, P. Papadopoulos, and B. Plenio, Phys. Rev. A **62,** 052317 (2000).
13. D. Collins, N. Linden, and S. Popescu, Preprint quant-ph/0005102 (2000).
14. Y. Z. Zheng, Y. J. Gu, L. B. Chen, and G. C. Guo, Chin. Phys. **11,** 529 (2002).
15. L. B. Chen, Chin. Phys. **11,** 881 (2002).
16. S. F. Huelga, J. A. Vaccaro, and A. Chefles, Phys. Rev. A **63,** 042303 (2001).
17. C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53,** 2046 (1996).
18. H. J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81,** 5932 (1998).
19. W. Dür, H. J. Briegel, J. I. Cirac, and P. Zoller, Phys. Rev. A **59,** 169 (1999).