The geometry of violation of Bell's inequality

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The purpose of this paper is to deduce an analytical expression for the violation of Bell's inequality by quantum theory and plane trigonometry, and expound the violation and maximal violation of the first, second type Bell's inequality in detail. Further, we find out the sufficient conditions for the region in which Bell's inequalities are violated.

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The topic of violation of Bell's inequality in quantum theory has been studied by authors[1−6] mainly from the point of operator algebra in Hilbert space. But the geometry involved in the violation of Bell's inequality has not been expounded. The purpose of this paper is to deduce an analytical expression for the violation of Bell's inequality by quantum theory and plane trigonometry, and expound the violation and maximal violation of the first, second type Bell's inequality in detail. Further, we find out the sufficient conditions for the region in which Bell's inequalities are violated. We note that the paper by Greenberger, Horne and Zeilinger (GHZ) has demonstrated Bell's theorem in a new way[7], by analyzing the geometry of a system consisting of three or more correlated spin $-1/2$ particles. Unlike Bell's original theorem and variants of it, GHZ's demonstration of the incompatibility of quantum mechanics with EPR's propositions concerns only perfect correlations rather than statistical correlations, and it completely dispenses with inequalities. The problems of geometry involved in more than three particles correlation are certainly more complicated and not discussed here.

For a better understanding of Bell's theorem, a brief review of the historical background is needed[8−14]. Einstein *et al.* (hereafter referred as EPR) presented that certain plausible propositions about locality, reality, and theoretical completeness are incompatible with the predictions of two-particle quantum mechanics[12,13]. They considered a system consisting of two spatially separated but quantum mechanically correlated particles. For this system, they showed that the results of various experiments of the associated system are predetermined, but this fact is not part of the quantum-mechanical description of the associated system. Hence that description is an incomplete one. To complete the description, it is necessary to postulate additional 'hidden variable'. In 1965, J. S. Bell considered a Gedanken-experiment of Bohm (a variant of that of EPR)[1,14]. The system consists of two spin $-1/2$ particles, prepared in the quantum-mechanical single state Ψ. Let $A_{\vec{a}}$ be the result of a measurement of the spin component of particle 1 of the pair along the direction of vector $\vec{a}$, and $B_{\vec{b}}$ be that of particle 2 along the direction of vector $\vec{b}$, we take the unit of spin as $\hbar/2$, hence $A_{\vec{a}}, B_{\vec{b}} = \pm 1$. The product $A_{\vec{a}} \cdot B_{\vec{b}}$ is a single observable of the two-particle system. For Gedanken-experiment, one can calculate the quantum-mechanical prediction for the expectation value of this observable

$$[E(\vec{a},\vec{b})]_{\Psi} = \langle \Psi | \vec{\sigma}_1 \cdot \vec{a} \vec{\sigma}_2 \cdot \vec{b} | \Psi \rangle = -\vec{a} \cdot \vec{b}. \quad (1)$$

In a deterministic hidden-variable theory the observable $A_{\vec{a}}$, $B_{\vec{b}}$ has a definite value $A_{\vec{a}}(\lambda) B_{\vec{b}}(\lambda)$. The expectation of $A_{\vec{a}}$, $B_{\vec{b}}$ for such theories is given by

$$E(\vec{a},\vec{b}) = \int_{\Lambda} A_{\vec{a}}(\lambda) B_{\vec{b}}(\lambda) \mathrm{d}\rho. \quad (2)$$

Subsequently Bell proved the first inequality

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| \le |1 + E(\vec{b},\vec{c})|. \quad (3)$$

By taking $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar, with $\vec{c}$ making an angle of $\frac{2\pi}{3}$ with $\vec{a}$, and $\vec{b}$ making an angle of $\frac{\pi}{3}$ with $\vec{a}$ and $\vec{c}$, it is shown that quantum-mechanical prediction and inequality (3) are incompatible with each other. In 1971 Bell proved the second inequality

$$-2 \le E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}') + E(\vec{a}',\vec{b}) + E(\vec{a}',\vec{b}') \le 2. \quad (4)$$

Similarly by taking $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ to be coplanar, and assuming the angles ϕ between vectors as

$$|\vec{a} - \vec{b}| = |\vec{a}' - \vec{b}| = |\vec{a}' - \vec{b}'| = \frac{1}{3} |\vec{a} - \vec{b}'| = \phi, \quad (5)$$

the incompatibility of quantum mechanical prediction with inequality (4) can be proved. Clauser and Horne also proved Bell's theorem for general local realistic theories, the results are formulated in terms of single and coincidence counts, rather than the expectation value considered in the ideal Gedanken-experiment[9]. This has the advantage that can be checked by measuring the frequency of joint detection of photons with polarisers in only two different relative orientations, yet in Clauser and Horne theory, taking vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ being coplanar and the angles ϕ between vectors satisfying Eq. (5) are the same as above used in the proof of the incompatibility of quantum-mechanical prediction and inequality. On account of this, the arrangement of this paper is as following. First, we deduce the violation of Bell's inequality by quantum theory under two conditions: one is the case of observation direction vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ being coplanar, other is that $\vec{a}$, $\vec{b}$, $\vec{a}'$ and $\vec{b}'$ being coplanar, then

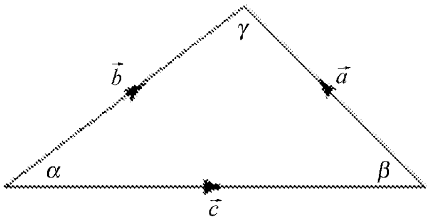

Fig. 1. The violation of the first Bell's inequality by quantum theory in case $\vec{a}$, $\vec{b}$ and $\vec{c}$ being coplanar.

we deduce the violation of Bell's inequality by quantum theory in general cases, i.e. without the restriction of vectors being coplanar.

Using the condition of the vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ being coplanar, and forming a triangle as shown in Fig. 1, the angles are denoted by α, β, γ, and the corresponding sides subtended are denoted by a, b, c. Based on plane trigonometry, the following identity establishes

$$\begin{aligned}&\cos\alpha+\cos\beta+\cos\gamma\\&=1+\frac{1}{2abc}(a+b-c)(a-b+c)(b+c-a)\\&=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.\end{aligned}\tag{6}$$

Noting the vectors marked in Fig. 1 and using the expectation value of quantum mechanics Eq. (1), we have

$$\begin{aligned}E(\vec{b},\vec{c})_\Psi&=-(\vec{b},\vec{c})=-\cos\alpha,\\E(\vec{a},\vec{c})_\Psi&=-(\vec{a},\vec{c})=\cos\beta,\\E(\vec{a},\vec{b})_\Psi&=-(\vec{a},\vec{b})=-\cos\gamma.\end{aligned}\tag{7}$$

Substituting Eq. (7) into (6), yields

$$\begin{aligned}1+E(\vec{b},\vec{c})_\Psi&=E(\vec{a},\vec{c})_\Psi-E(\vec{a},\vec{b})_\Psi\\&\quad-\frac{1}{2abc}(a+b-c)(b+c-a)(c+a-b)\\&<E(\vec{a},\vec{c})_\Psi-E(\vec{a},\vec{b})_\Psi,\end{aligned}\tag{8}$$

and

$$|E(\vec{a},\vec{c})_\Psi-E(\vec{a},\vec{b})_\Psi|>1+E(\vec{b},\vec{c})_\Psi.\tag{9}$$

In comparison Eq. (9) with the first Bell's inequality (3), we have

$$|E(\vec{a},\vec{b})-E(\vec{a},\vec{c})|\le 1+E(\vec{b},\vec{c}).\tag{10}$$

We conclude that the first Bell's inequality is violated in the case of vectors shown in Fig. 1. As for the violation of the second Bell's inequality, let us consider the triangles formed by the coplanar vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ shown in Fig. 2 and write the expectation value of quantum mechanics as

$$\begin{aligned}E(\vec{a},\vec{b})_\Psi&=-\cos\gamma,\\E(\vec{a}',\vec{b})_\Psi&=-\cos\gamma',\\E(\vec{a},\vec{b}')_\Psi&=\cos\beta,\\E(\vec{a}',\vec{b}')_\Psi&=-\cos\beta'.\end{aligned}\tag{11}$$

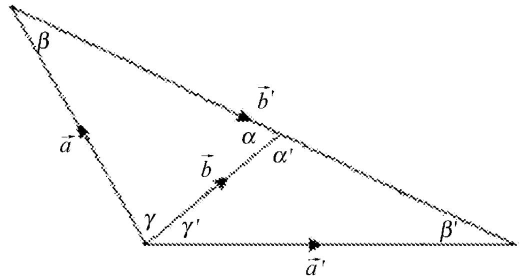

Fig. 2. The violation of the second Bell's inequality by quantum theory in case $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ being coplanar.

$$\begin{aligned}&E(\vec{a},\vec{b})_\Psi-E(\vec{a},\vec{b}')_\Psi+E(\vec{a}',\vec{b})_\Psi+E(\vec{a}',\vec{b}')_\Psi\\&=-\cos\gamma-\cos\beta-\cos\gamma'-\cos\beta'\\&=-(\cos\alpha+\cos\beta+\cos\gamma+\cos\alpha'+\cos\beta'+\cos\gamma')\\&=-(1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\\&\quad+1+4\sin\frac{\alpha'}{2}\sin\frac{\beta'}{2}\sin\frac{\gamma'}{2}).\end{aligned}\tag{12}$$

$$\begin{aligned}&|E(\vec{a},\vec{b})_\Psi-E(\vec{a},\vec{b}')_\Psi+E(\vec{a}',\vec{b})_\Psi+E(\vec{a}',\vec{b}')_\Psi|\\&=2+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+4\sin\frac{\alpha'}{2}\sin\frac{\beta'}{2}\sin\frac{\gamma'}{2}\\&\ge 2.\end{aligned}\tag{13}$$

Obviously Eq. (13) is a violation of the second Bell's inequality (4)

$$|E(\vec{a},\vec{b})_\Psi-E(\vec{a},\vec{b}')_\Psi+E(\vec{a}',\vec{b})_\Psi+E(\vec{a}',\vec{b}')_\Psi|<2.\tag{14}$$

The violation of the first and second Bell's inequalities can be written from Eqs. (9) and (13) in the form

$$\begin{aligned}V_1&=|E(\vec{a},\vec{b})_\Psi-E(\vec{a},\vec{c})_\Psi|-|1+E(\vec{b},\vec{c})_\Psi|\\&=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2},\end{aligned}\tag{15}$$

$$\begin{aligned}V_2&=|E(\vec{a},\vec{b})_\Psi-E(\vec{a},\vec{b}')_\Psi+E(\vec{a}',\vec{b})_\Psi+E(\vec{a}',\vec{b}')_\Psi|-2\\&=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+4\sin\frac{\alpha'}{2}\sin\frac{\beta'}{2}\sin\frac{\gamma'}{2}.\end{aligned}\tag{16}$$

Obviously for $\alpha=\beta=\gamma=\frac{\pi}{3}$, Eq. (15) yields the maximal violation of the first Bell's inequality $V_{1\mathrm{m}}=1/2$, and for $\alpha=\alpha'=\frac{\pi}{2}$, $\beta=\beta'=\gamma=\gamma'=\frac{\pi}{4}$, Eq. (16) yields $V_{2\mathrm{m}}=8\sin\frac{\pi}{4}\sin^2\frac{\pi}{8}=2\sqrt{2}-2$.

Now we prove the violation of the first Bell's inequality by quantum theory in general case of vectors $\vec{a}_s$, $\vec{b}$ and $\vec{c}$ being not in a plane. Without loss of generality, taking $\vec{b}$ as the polar axis, writing the polar angles of $\vec{a}_s$, $\vec{c}$ as $\vec{a}_s(\gamma,\varphi)$, $\vec{c}(\alpha,0)$ and rotating $\vec{a}_s$ around $\vec{b}$ with the angle γ invariant up to $\vec{a}(\gamma,\pi)$ such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ being coplanar, shown in Fig. 3, and denoting the angles between $\vec{a}_s$ and $\vec{c}$, $\vec{a}$ and $\vec{c}$ by $\pi-\beta_s$, $\pi-\beta$ respectively, finally we have

$$\begin{aligned}\cos(\vec{a}_s\cdot\vec{c})&=\cos\gamma\cos\alpha+\sin\alpha\sin\gamma\cos\varphi\\&=\cos(\gamma+\alpha)+\sin\alpha\sin\gamma(1+\cos\varphi),\end{aligned}$$

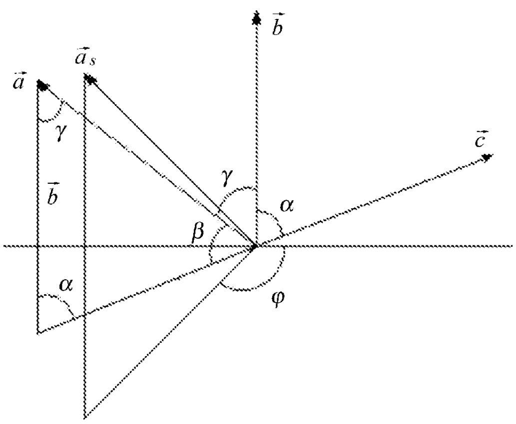

Fig. 3. The violation of the second Bell's inequality by quantum theory in general case.

$$\cos(\pi-\beta_s)=\cos(\pi-\beta)+\sin\alpha\sin\gamma(1+\cos\varphi),$$
$$\cos\alpha+\cos\beta_s+\cos\gamma=\cos\alpha+\cos\beta+\cos\gamma$$
$$-\sin\alpha\sin\gamma(1+\cos\varphi). \quad (17)$$

Substituting Eq. (6) into (17), yields

$$\begin{aligned}&\cos\alpha+\cos\beta_s+\cos\gamma\\&=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}-\sin\alpha\sin\gamma(1+\cos\varphi)\\&=1+4\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}[\cos\frac{\gamma+\alpha}{2}-\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}(1+\cos\varphi)]\\&=1-\sin\alpha\sin\gamma[\cos\varphi+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}]. \qquad (18)\end{aligned}$$

Referring to Fig. 1 with $\vec{a}$ replaced by $\vec{a}_s$ and writing

$$\begin{aligned}E(\vec{b},\vec{c})_\Psi&=-(\vec{b},\vec{c})=-\cos\alpha,\\E(\vec{a}_s,\vec{c})_\Psi&=-(\vec{a}_s,\vec{c})=\cos\beta_s,\\E(\vec{a}_s,\vec{b})_\Psi&=-(\vec{a}_s,\vec{b})=-\cos\gamma. \qquad (19)\end{aligned}$$

Substituting Eqs. (19) into (18), leads to

$$\begin{aligned}&E(\vec{a}_s,\vec{c})_\Psi-E(\vec{a}_s,\vec{b})_\Psi\\&=1+E(\vec{b},\vec{c})_\Psi-\sin\alpha\sin\gamma[\cos\varphi+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}]. \quad (20)\end{aligned}$$

Therefore, depending on the signature of $[\,]=[\cos\varphi+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}]<0$ or >0, we designate the regions in which the first Bell's inequality is violated or not by quantum theory.

$$\begin{aligned}&|E(\vec{a}_s,\vec{b})_\Psi-E(\vec{a}_s,\vec{c})_\Psi|>1+E(\vec{b},\vec{c})_\Psi,\\&\quad[\,]<0,\ \text{violated},\\&|E(\vec{a}_s,\vec{b})_\Psi-E(\vec{a}_s,\vec{c})_\Psi|<1+E(\vec{b},\vec{c})_\Psi,\\&\quad[\,]>0,\ \text{not violated}. \qquad (21)\end{aligned}$$

As for the second Bell's inequality in general case, we refer to Fig. 2. Taking $\vec{b}$ as the polar axis, the vectors $\vec{a}(\gamma,\pi)$, $\vec{a}'(\gamma',\pi)$ being replaced by $\vec{a}_s(\gamma,\varphi)$, $\vec{a}'_s(\gamma',\varphi')$, in the same way we can deduce

$$\begin{aligned}&E(\vec{a}_s,\vec{b})-E(\vec{a}_s,\vec{b}')+E(\vec{a}'_s,\vec{b})+E(\vec{a}'_s,\vec{b}')\\&=-(1-\sin\beta\sin\gamma[\cos\varphi+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}]\\&\quad+1-\sin\beta'\sin\gamma'[\cos\varphi'+\tan\frac{\beta'}{2}\tan\frac{\gamma'}{2}])\\&=-2+\{\,\}, \qquad (22)\end{aligned}$$

where $\{\,\}=\sin\beta\sin\gamma[\cos\varphi+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}]+\sin\beta'\sin\gamma'[\cos\varphi'+\tan\frac{\beta'}{2}\tan\frac{\gamma'}{2}]$. Depending on $\{\,\}<0$ or >0, we can designate the regions in which the second Bell's inequality is violated or not by quantum theory.

$$\begin{aligned}&|E(\vec{a}_s,\vec{b})_\Psi-E(\vec{a}_s,\vec{b}')_\Psi+E(\vec{a}'_s,\vec{b})_\Psi+E(\vec{a}'_s,\vec{b}')_\Psi|>2,\\&\quad\{\,\}<0,\ \text{violated},\\&|E(\vec{a}_s,\vec{b})_\Psi-E(\vec{a}_s,\vec{b}')_\Psi+E(\vec{a}'_s,\vec{b})_\Psi+E(\vec{a}'_s,\vec{b}')_\Psi|<2,\\&\quad\{\,\}>0,\ \text{not violated}. \qquad (23)\end{aligned}$$

By varying the angles α, γ with β invariant and $\alpha+\beta+\gamma=\pi$, such that $\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}$ attains a maximum $\tan^2\frac{\pi-\beta}{4}$ and consequently the region for the first Bell's inequality violated by quantum theory can be designated as

$$[\,]=\cos\varphi+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}<\cos\varphi+\tan^2\frac{\pi-\beta}{4}<0. \quad (24)$$

The condition (24) $\cos\varphi<-\tan^2\frac{\pi-\beta}{4}$ (Fig. 4(a) dash region) is a sufficient condition for the first Bell's inequality violated by quantum theory. Using the same condition for the second Bell's inequality violated by quantum theory, we have

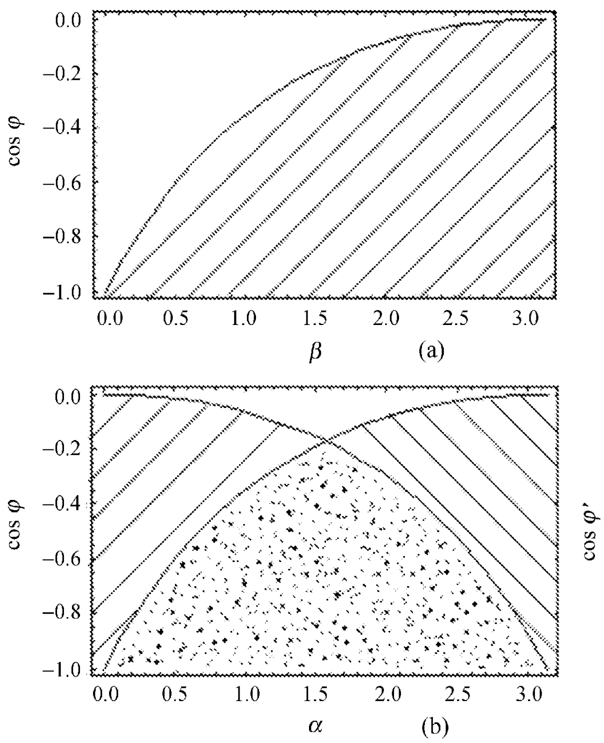

Fig. 4. The sufficient condition for the first (a, dash region) and second (b, dot region) Bell's inequality violated by quantum theory.

$$\begin{aligned}\{\ \} &= \sin\beta\sin\gamma[\cos\varphi + \tan\frac{\beta}{2}\tan\frac{\gamma}{2}] \\ &\quad + \sin\beta'\sin\gamma'[\cos\varphi' + \tan\frac{\beta'}{2}\tan\frac{\gamma'}{2}] \\ &\le \sin\beta\sin\gamma[\cos\varphi + \tan^2\frac{\pi-\alpha}{4}] \\ &\quad + \sin\beta'\sin\gamma'[\cos\varphi' + \tan^2\frac{\pi-\alpha'}{4}] \\ &= \sin\beta\sin\gamma[\cos\varphi + \tan^2\frac{\pi-\alpha}{4}] \\ &\quad + \sin\beta'\sin\gamma'[\cos\varphi' + \tan^2\frac{\alpha}{4}] \\ &< 0. \end{aligned} \quad (25)$$

Obviously the conditions $\cos\varphi < -\tan^2\frac{\pi-\alpha}{4}$ and $\cos\varphi' < -\tan^2\frac{\alpha}{4}$ (Fig. 4(b) dot region) guarantee the second Bell's inequality violated.

In conclusion, under the conditions of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$; $\vec{a}$, $\vec{b}$, $\vec{a}'$ and $\vec{b}'$ being coplanar, according to quantum mechanics and plane geometry we derive the analytical expressions of the violation and maximal violation of Bell's first and second inequalities as Eqs. (9), (13); (15), (16). The violation of the first Bell's inequality $V_{1\text{m}}$ ranges from 0 to 1/2. The violation of the second Bell's inequality $V_{2\text{m}}$ ranges from 0 to $2\sqrt{2}-2$, the maximum $V_{2\text{m}}+2 = 2\sqrt{2}$ is allowed by Cirel'son's theorem[3,4]. In general case, relaxing the restriction of vectors being coplanar, we find out sufficient conditions for the region in which the Bell's inequalities are violated.

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