Exact expression for decoherence factor in the time-dependent generalized Cini model

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The present letter finds the complete set of exact solutions of the time-dependent generalized Cini model by making use of the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation and, based on this, the general explicit expression for the decoherence factor is therefore obtained. This study provides us with a useful method to consider the geometric phase and topological properties in the time-dependent quantum decoherence process.

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Quantum decoherence problem is one of the most important and interesting topics in quantum mechanics, quantum optics and quantum information[1,2]. Solvable models in quantum mechanics enable one to investigate quantum measurement problems rather conveniently[3−5]. A good number of authors[6−9] have studied some useful models such as Hepp-Coleman model[10] and Cini model[11]. The exact solvability of these models often provides physicists with a clear understanding of the physical phenomena involved and yields rich physical insights[12]. In this work, the first important step is to obtain the exact solutions of the Schrödinger equation and the time-evolution operator that can be applied to the calculation of the decoherence factor and study of the wavefunction collapse, etc. Although the exact solutions and the decoherence of these models have been extensively investigated by many authors in the literature[6−11], the coefficients and parameters in these Hamiltonians are merely time-independent (or partially time-dependent), to the best of our knowledge. In the present letter, we obtain the explicit time-evolution operator and the decoherence factor of the totally time-dependent generalized Cini model, where all the parameters depend on time. Since the time-dependent quantum model possesses more rich properties, e.g., geometric phase and topological feature[4,5], it is of essential significance to consider the time-dependent case of quantum models.

Time-dependent system is governed by the time-dependent Schrödinger equation. The invariant theory suggested by Lewis and Riesenfeld in 1969 can solve the time-dependent Schrödinger equation[13]. In 1991, Gao *et al.* proposed a generalized invariant theory[14,15], by introducing basic invariants, which enable one to find the complete set of commuting invariants for some time-dependent multi-dimensional systems[16−18]. Since the time-dependent case and geometric phase factor has never been considered in the time-dependent decoherence process, we will analyze the generalized Cini model in what follows and then calculate the time-dependent decoherence factor by making use of these invariant theories.

The original Cini model for the correlation between the states of the measured system and the measuring instrument-detector is built for a two-level system interacting with the detector. Liu and Sun generalized this Cini model to an M-level system[19]. The Hamiltonian of this generalized model is

$$H = H_{\rm S} + H_{\rm D} + H_{\rm I}, \tag{1}$$

where $H_{\rm S}$ is the model Hamiltonian of the measured system S with M levels and $H_{\rm D}$ is the free Hamiltonian of the two-boson-state detector D. They are generally of the forms

$$H_{\rm S} = \sum_{k=1}^{M} E_k \left|\Phi_k\right\rangle \left\langle\Phi_k\right|,$$
$$H_{\rm D} = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 \tag{2}$$

with the creation and annihilation operators $a_i^\dagger$, a_i satisfying the commuting relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{3}$$

The interaction Hamiltonian $H_{\rm I}$ is given by

$$H_{\rm I} = \sum_n \left|\Phi_n\right\rangle \left\langle\Phi_n\right| (g_n a_1^\dagger a_2 + g_n^* a_2^\dagger a_1). \tag{4}$$

In this letter, all the coefficients such as E_k, ω_1, ω_2, g_n and g_n^* in the Hamiltonian are time-dependent, and the Schrödinger equation of the time-dependent generalized Cini model may be written as

$$H(t)\left|\Psi(t)\right\rangle_s = i\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle_s. \tag{5}$$

It can be seen from the form of the Hamiltonian that both $\left|\Phi_k\right\rangle\left\langle\Phi_k\right|$ and $N = \frac{a_1^\dagger a_1 + a_2^\dagger a_2}{2}$ commute with H, namely, $[\left|\Phi_k\right\rangle\left\langle\Phi_k\right|, H] = [N, H] = 0$. Hence, a generalized quasialgebra which enables one to obtain the complete set of exact solutions of the Schrödinger equation can be found by working in a sub-Hilbert-space corresponding to the particular eigenvalues of both $|\Phi_k\rangle\langle\Phi_k|$ and N. In order to use the Lewis-Riesenfeld invariant theory[13], we take $J_+ = a_1^\dagger a_2$, $J_- = a_2^\dagger a_1$, $J_3 = \frac{a_1^\dagger a_1 - a_2^\dagger a_2}{2}$, which satisfy the commuting relations $[J_3, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_3$. In this sub-Hilbert-space the Hamiltonian can therefore be rewritten as

$$H_{n,k}(t) = E_k + g_k J_+ + g_k^* J_- \\ +(\omega_1 - \omega_2)J_3 + n(\omega_1 + \omega_2), \tag{6}$$

with n being the eigenvalue of N and satisfying

$$N|n_1, n_2\rangle = n|n_1, n_2\rangle, \quad n = \frac{1}{2}(n_1 + n_2). \tag{7}$$

Thus in the sub-Hilbert-space we write the Schrödinger equation in the form

$$H_{n,k}(t)|\Psi_{n,k}(t)\rangle_s = i\frac{\partial}{\partial t}|\Psi_{n,k}(t)\rangle_s, \tag{8}$$

and $|\Psi(t)\rangle_s$ can be obtained from

$$|\Psi(t)\rangle_s = \sum_n \prod_k c_{n,k}|\Psi_{n,k}(t)\rangle_s |\Phi_k\rangle, \tag{9}$$

where $c_{n,k}$ is time-independent and determined by the initial conditions.

For the sake of using the invariant theory conveniently, we rewrite the Hamiltonian $H_{n,k}(t)$ as follows

$$H_{n,k}(t) = c_k(t)\{\frac{1}{2}\sin\theta_k(t)\exp[-i\varphi_k(t)]J_+ \\ +\frac{1}{2}\sin\theta_k(t)\exp[i\varphi_k(t)]J_- + \cos\theta_k(t)J_3\} \\ +f_{n,k}(t), \tag{10}$$

where

$$c_k(t)\cos\theta_k(t) = \omega_1 - \omega_2, \\ f_{n,k}(t) = E_k + n(\omega_1 + \omega_2), \\ \frac{1}{2}c_k(t)\sin\theta_k(t)\exp[-i\varphi_k(t)] = g_k. \tag{11}$$

In accordance with the invariant theory, an invariant that satisfies invariant equation

$$\frac{\partial I_k(t)}{\partial t} + \frac{1}{i}[I_k(t), H_{n,k}(t)] = 0 \tag{12}$$

should be constructed. It follows from Eq. (12) that the invariant $I_k(t)$ is the linear combination of $J_\pm$ and J_3 and may be written as

$$I_k(t) = \frac{1}{2}\sin\lambda_k(t)\exp[-i\gamma_k(t)]J_+ \\ +\frac{1}{2}\sin\lambda_k(t)\exp[i\gamma_k(t)]J_- + \cos\lambda_k(t)J_3. \tag{13}$$

Using the invariant equation (12), two auxiliary equations by which $\lambda_k(t)$ and $\gamma_k(t)$ will be determined can be derived

$$\dot\lambda_k(t) = c_k\sin\theta_k\sin(\varphi_k - \gamma_k), \\ \dot\gamma_k(t) = c_k[\cos\theta_k - \sin\theta_k\cot\lambda_k\cos(\varphi_k - \gamma_k)]. \tag{14}$$

In order to obtain the exact solution of the time-dependent Schrödinger equation (8), we introduce an invariant-related unitary transformation operator $V_k(t)$

$$V_k(t) = \exp[\beta_k(t)J_+ - \beta_k^*(t)J_-], \tag{15}$$

where the time-dependent parameter

$$\beta_k(t) = -\frac{\lambda_k(t)}{2}\exp[-i\gamma_k(t)], \\ \beta_k^*(t) = -\frac{\lambda_k(t)}{2}\exp[i\gamma_k(t)]. \tag{16}$$

$V_k(t)$ can be readily shown to transform the time-dependent invariant $I_k(t)$ into I_{kV} that is time-independent

$$I_{kV} \equiv V_k^\dagger(t)I_k(t)V_k(t) = J_3. \tag{17}$$

The eigenstate of the $I_{kV} = J_3$ corresponding to the eigenvalue m is denoted by $|j,m\rangle$, where

$$|j,m\rangle = \sum_{n_1 n_2}\langle n_1, n_2|j,m\rangle|n_1, n_2\rangle, \\ m = \frac{1}{2}(n_1 - n_2). \tag{18}$$

By making use of $V_k(t)$ in Eq. (15) and the Baker-Campbell-Hausdorff formula[20]

$$V^\dagger(t)\frac{\partial}{\partial t}V(t) = \frac{\partial}{\partial t}L + \frac{1}{2!}[\frac{\partial}{\partial t}L, L] + \frac{1}{3!}[[\frac{\partial}{\partial t}L, L], L] \\ +\frac{1}{4!}[[[\frac{\partial}{\partial t}L, L], L], L] + \cdots, \tag{19}$$

where $V(t) = \exp[L(t)]$, one can obtain $H_{n,kV}(t)$ from $H_{n,k}(t)$,

$$H_{n,kV}(t) = V_k^\dagger(t)H_{n,k}(t)V_k(t) - V_k^\dagger(t)i\frac{\partial V_k(t)}{\partial t} \\ = \{[\cos\lambda_k\cos\theta_k + \sin\lambda_k\sin\theta_k\cos(\gamma_k - \varphi_k)] \\ +\dot\gamma_k(1 - \cos\lambda_k)\}J_3 + f_{n,k}. \tag{20}$$

From the two expressions (17) and (20), one can see that $H_{n,kV}(t)$ differs from I_{kV} only by a time-dependent c-number factor and $f_{n,k}$. Use is made of the invariant-related unitary transformation formulation and the general solution of the time-dependent Schrödinger equation (8) is therefore obtained,

$$|\Psi_{n,k}(t)\rangle_s = \sum_m C_m(n,k) \\ \cdot\exp[i\phi_m(n,k,t)]V_k(t)|j,m\rangle, \tag{21}$$

with the coefficients $C_m = \langle j,m,t=0|\Psi_{n,k}(0)\rangle_s$. The phase $\phi_m(n,k,t) = \phi_m^{(d)}(n,k,t) + \phi_m^{(g)}(k,t)$ includes the dynamical phase

$$\phi_m^{(d)}(n,k,t) = -\int_0^t \langle j,m|V_k^\dagger(t')H_{n,k}(t')V_k(t')|j,m\rangle\,dt' \\ = -m\int_0^t\{\cos\lambda_k(t')\cos\theta_k(t') \\ +\sin\lambda_k(t')\sin\theta_k(t')\cos[\gamma_k(t') - \varphi_k(t')] \\ +\frac{1}{m}f_{n,k}(t')\}dt', \tag{22}$$

and the geometric phase

$$\phi_m^{(g)}(k,t) = -\int_0^t \langle m| - V_k^\dagger(t')i\frac{\partial V_k(t')}{\partial t'}|m\rangle \, dt' = -m\int_0^t \dot{\gamma}_k(t')[1-\cos\lambda_k(t')]dt'. \quad (23)$$

It follows from Eq. (23) that if the parameter $\dot{\gamma}_k$ is taken to be time-independent, then the geometric phase in one cycle ($T = \frac{2\pi}{\dot{\gamma}_k}$) is $\phi_m^{(g)}(k,T) = -m[2\pi(1-\cos a)]$, where $2\pi(1-\cos m)$ is an expression for the solid angle over the parameter space of the invariant. It is of interest that $-m[2\pi(1-\cos a)]$ is equal to the magnetic flux produced by a magnetic monopole (and the gravitomagnetic monopole) of strength $\frac{\lambda}{4\pi m}$ existing at the origin of the parameter space[21]. This, therefore, implies that geometric phase differs from dynamical phase and it involves the global and topological properties of the time evolution of a quantum system.

Since we have exact solutions of the time-dependent Cini model, we can consider the decoherence factor that is given by

$$F_{k,l}(t) = \langle j,m| V_k^\dagger(t)V_l(t)|j,m\rangle. \quad (24)$$

Further calculation yields

$$F_{k,l}(t) = \exp[-m(\beta_l\beta_k^* - \beta_k\beta_l^*)]\langle j,m|\exp[(\beta_l-\beta_k)J_+ - (\beta_l^*-\beta_k^*)J_-]|j,m\rangle, \quad (25)$$

which is the general expression for the decoherence factor of the time-dependent Cini model. Although the expression (25) is somewhat complicated, it is just the general explicit expression that does not contain the chronological product. To show that the decoherence factor (25) can reduce to the familiar results in the time-independent or partially time-dependent Cini model, we consider a special and simple case where the Hamiltonian is[22]

$$H_{n,k} = c_k J_2 + f_{n,k} \quad (26)$$

with $J_2 = \frac{1}{2i}(J_+ - J_-)$. It follows from Eq. (26) that, in this case, $\varphi_k = \frac{\pi}{2}$, $\omega_1 = \omega_2$, $\sin\theta_k = 1$. One can therefore arrive at

$$\gamma_k = 0, \quad \dot{\lambda}_k = c_k, \quad \beta_k = \beta_k^* = -\frac{\lambda_k}{2}. \quad (27)$$

In the similar fashion, we have

$$\gamma_l = 0, \quad \dot{\lambda}_l = c_l, \quad \beta_l = \beta_l^* = -\frac{\lambda_l}{2}. \quad (28)$$

It is verified with the help of Eq. (25) that

$$F_{k,l}(t) = \langle j,m|\exp[i\int_0^t (c_k - c_l)dt' J_2]|j,m\rangle. \quad (29)$$

If the state of the measuring instrument-detector at the initial $t = 0$ is $|j,j\rangle$, then the decoherence factor is

$$F_{k,l}(t) = \langle j,j|\exp[i\int_0^t (c_k - c_l)dt' J_2]|j,j\rangle = [\cos(\int_0^t \frac{c_k - c_l}{2}dt')]^{2j}. \quad (30)$$

Note that the expression (30) is a well-known result and in complete agreement with the one obtained by Sun *et al.*[22]. When $j \to \infty$ and $\int_0^t \frac{c_k-c_l}{2}dt' \neq n\pi$ ($n = 0, \pm1, \pm2, \cdots$), $F_{k,l}(t) \to 0$, which means the wavefunction collapse occurs under the classical limit.

To conclude this letter, we briefly discuss the concepts of the exact solution and the explicit solution. The expression (21) is a particular exact solution corresponding to the particular eigenvalue m of the invariant and thus the general solutions of the time-dependent Schrödinger equation are easily obtained by using the linear combinations of all these particular solutions. Generally speaking, in Quantum Mechanics, solution with chronological-product operator (time-order operator) P is often called the formal solution. In the present letter, however, the solution of the Schrödinger equation governing a time-dependent system is sometimes called the explicit solution, for reasons that the solution does not involve time-order operator. But, on the other hand, by using Lewis-Riesenfeld invariant theory, there always exist time-dependent parameters, for instance, $\lambda_k(t)$ and $\gamma_k(t)$ in this letter, which are determined by the auxiliary Eq. (14). According to the traditional practice, when employed in experimental analysis and compared with experimental results, these nonlinear auxiliary equations should be solved often by means of numerical calculation. From above viewpoints, the concept of explicit solution is understood in a relative sense, namely, it can be considered explicit solution when compared with the time-evolution operator $U(t) = P\exp[\frac{1}{i}\int_0^t H(t')dt']$ involving time-order operator, P; whereas, it cannot be considered completely explicit solution for it is expressed in terms of some time-dependent parameters which should be obtained via the auxiliary equations. Hence, conservatively speaking, we regard the solution of the time-dependent system presented in the letter as exact solution rather than as explicit solution.

The present letter obtains exact solutions and decoherence factor of the time-dependent Cini model by working in the sub-Hilbert space corresponding to the eigenvalue of two invariants, which commute with the Hamiltonian, and by using the invariant-related unitary transformation method. The invariant-related unitary transformation formulation is an effective method for treating time-dependent problems[23]. This formulation replaces eigenstates of the time-dependent invariants with those of the time-independent invariants through the unitary transformation. It uses the invariant-related unitary transformation and obtains the explicit expression for the time-evolution operator, instead of the formal solution associated with the chronological product. In view of what has been discussed above, it can be seen that the invariant theory is appropriate to treat the time-dependent quantum decoherence. Apparently, this method is easy to generalize to the time-dependent Hepp-Coleman model[10]. Since the geometric phase factor appears in the time-dependent systems, it is interesting to consider the geometric phase in the time-dependent quantum decoherence.

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