Improving the fidelity of continuous-variable quantum teleportation by tuning displacement gain

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The fidelity of teleportation of continuous quantum variables can be improved by tuning the local displacement gain. We investigate the optimization of the fidelity for the teleportation of Schrödinger cat states, and of coherent states. It is found that the gain corresponding to the maximum fidelity is not equal to one for the two input states in the case of the small squeezing degree of the entanglement resource, while unity displacement gain is the best choice for teleporting arbitrary quantum states in the case of large squeezing.

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Quantum teleportation, initially suggested by Bennett *et al*[1]., is a process that transmits an unknown qubit from a sender to a distant receiver via an entangled quantum channel with the help of some classical information. Subsequently, Vaidman[2], Braunstein and Kimble[3] extended the concept to the teleportation of continuous quantum variables corresponding to states of infinite dimensional systems. Moreover, the teleportation of optical coherent states was realized experimentally by means of the entanglement of a two-mode squeezed vacuum[4]. Because of the high detection efficiency of the homodyne measurement and convenient manipulation of continuous variable states, much of people's attention has been attracted to the study of continuous quantum variables, such as quantum dense coding[5,6], entanglement swapping[7,8], quantum error correction[9], quantum cryptography[10], quantum computation[11], entanglement purification[12], and quantum cloning[13].

It is well-known that fidelity is an important quality in the subject of quantum information[14]. Fidelity can measure the performance of quantum teleportation and describe the similarity between the input state and the teleported state. In order to obtain better output results, people have tried to increase the fidelity of quantum teleportation in continuous-variable systems using many methods[15−17]. Since the displacement gain is an extra parameter, it allows the maximization of the fidelity. In this paper, based on tuning the displacement gain in the transformation of the output field, we consider a protocol about the optimization of the fidelity of continuous-variable quantum teleportation. Firstly, we obtain a general expression of the gain-dependent output for the teleportation of any input states. Then, the two specific states, Schrödinger cats and coherent states, are applied to illustrate our protocol. Our results demonstrate that in the case of small squeezing of the entanglement resource, the optimal gain of local operation is not equal to one for the two input states. With the increase of the squeezing parameter, the optimal gain will become unity for teleporting arbitrary quantum states.

It is generally accepted that a two-mode squeezed vacuum state can be generated as the output of a nondegenerate optical parametric amplifier[18]. The Wigner function of such a squeezed state can be written as

$$W(\alpha_1;\alpha_2)=\frac{4}{\pi^2}\exp\{-\mathrm{e}^{-2r}[(x_1-x_2)^2+(p_1+p_2)^2] -\mathrm{e}^{2r}[(x_1+x_2)^2+(p_1-p_2)^2]\}, \quad (1)$$

where the complex quadrature phase variable $\alpha_j=x_j+ip_j$ $(j=1,2)$ and r is the degree of squeezing. For large squeezing parameter, $r\to\infty$ and the state (1) is given by

$$W(\alpha_1;\alpha_2)\longrightarrow W_{EPR}=C\delta(x_1+x_2)\delta(p_1-p_2), \quad (2)$$

which becomes an EPR state and exhibits the maximal entanglement. In this way, there is a strong correlation between x_1 and $-x_2$ as well as between p_1 and p_2.

Now we investigate a protocol for teleporting an original unknown state $\alpha_{\rm in}=x_{\rm in}+ip_{\rm in}$. Here, the real quantities $(x_{\rm in},p_{\rm in})$ correspond to quadrature amplitudes of the electromagnetic field. Suppose that a sender wants to teleport the unknown input state to a receiver by using the entanglement resource described by Eq. (1) as the quantum channel. At the beginning of the teleportation, the whole state of the system, is a product of the input state and the two-mode squeezed vacuum, and can be described by the total Wigner function

$$W_{\rm t}(\alpha_{\rm in};\alpha_1;\alpha_2)=W_{\rm in}(\alpha_{\rm in})W(\alpha_1;\alpha_2), \quad (3)$$

where $W_{\rm in}(\alpha_{\rm in})$ is the Wigner function of the input state. As in usual teleportation, the protocol comprises two operations at the sending station and one operation at the receiving station. At the sending station, the input state $\alpha_{\rm in}$ is coupled with a mode 1 of the quantum channel by 50/50 beam splitter $\beta_{a,b}=\frac{1}{\sqrt{2}}(\alpha_{\rm in}\pm\alpha_1)$. After the beam splitter, the total Wigner function can be written as

$$W_{\rm t}(\beta_a;\beta_b;\alpha_2)=W_{\rm t}(\frac{\beta_a+\beta_b}{\sqrt{2}};\frac{\beta_a-\beta_b}{\sqrt{2}};\alpha_2), \quad (4)$$

which manifests entanglement between the original input state and the entangled state consisting of the two-mode squeezed vacuum. Using homodyne detectors at the two output ports of the beam splitter, the real part of β_a and the imaginary part of β_b are simultaneously measured by choosing the phases of their respective local oscillators. We assume that the homodyne detectors provides an ideal quantum measurement of quadratures amplitudes, i.e., the detection efficiency is equal to one. The measurement results of the two conjugate quadrature variables, $(x_a, p_b) \equiv (\mathrm{Re}\beta_a = \frac{1}{\sqrt{2}}(x_{\mathrm{in}} + x_1), \mathrm{Im}\beta_b = \frac{1}{\sqrt{2}}(p_{\mathrm{in}} - p_1))$, are sent to the receiver through some classical means of communication. However, all information about the observables corresponding to the imaginary part of β_a and the real part of β_b is lost. Based on the classical results from the sender, the receiver performs a displacement operation in the mode 2 of the squeezed state,

$$\begin{aligned} x_2 &\to x_{\mathrm{out}} = x_2 + \lambda\sqrt{2}x_a, \\ p_2 &\to p_{\mathrm{out}} = p_2 + \lambda\sqrt{2}p_b, \end{aligned} \tag{5}$$

where the parameter λ is a displacement gain for the local operation.

After the transformations, the final teleported state of mode 2 at the receiver's side can be given by the Wigner function $W_{\mathrm{out}}(\alpha_{\mathrm{out}})$,

$$\begin{aligned} W_{\mathrm{out}}(\alpha_{\mathrm{out}}) &= \int \mathrm{d}x_a \mathrm{d}p_a \mathrm{d}x_b \mathrm{d}p_b \\ &\cdot W_{\mathrm{t}}(x_a, p_a, x_b, p_b, x_{\mathrm{out}} - \lambda\sqrt{2}x_a, p_{\mathrm{out}} - \lambda\sqrt{2}p_b) \\ &= \frac{1}{\lambda^2}\int \mathrm{d}^2\xi W_{\mathrm{in}}(\xi) G_\tau(\frac{\alpha_{\mathrm{out}}}{\lambda} - \xi) \\ &= \frac{1}{\lambda^2} W_{\mathrm{in}} \circ G_\tau(\frac{\alpha_{\mathrm{out}}}{\lambda}), \end{aligned} \tag{6}$$

where, $\circ$ denotes convolution and $G_\tau(\gamma) = \frac{1}{\pi\tau}\exp(-\frac{|\gamma|^2}{\tau})$ is a Gaussian distribution with

$$\tau = \frac{(1+\lambda^2)(\mathrm{e}^{2r} + \mathrm{e}^{-2r}) - 2\lambda(\mathrm{e}^{2r} - \mathrm{e}^{-2r})}{4\lambda^2}. \tag{7}$$

Equation (6) describes a general formula for the output of the teleportation of any input states $W_{\mathrm{in}}(\alpha_{\mathrm{in}})$. Note that for $\lambda = 1$, $\tau = \mathrm{e}^{-2r}$ and the Wigner function of the teleported state is of the same form as Eq. (4) in Ref. [3]. In the following, without loss of generality, two particular input states, a Schrödinger cat state and a coherent state, are given to demonstrate how to improve the teleportation fidelity by changing the displacement gain, respectively.

We first consider the teleportation of the input state prepared in a superposition of two coherent states,

$$|\phi\rangle = N(|\beta\rangle + \mathrm{e}^{i\varphi}|-\beta\rangle), \tag{8}$$

where $N = \sqrt{\frac{1}{2[1+\exp(-2|\beta|^2)\cos\varphi]}}$ is the normalization factor. Such macroscopic superposition states are regarded as Schrödinger cat states, which exhibit nonclassical properties because of the quantum interference between the two coherent components[19,20]. It deserves mentioning that for $\varphi = 0$, the Eq. (8) represents the even coherent state while an odd coherent state corresponds to $\varphi = \pi$. In general, the Wigner function $W(\alpha)$ for a state of the density operator ρ is defined as

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2}\int \mathrm{d}^2\eta \exp(\eta^*\alpha - \eta\alpha^*) \\ &\cdot \mathbf{Tr}[\rho\exp(\eta\hat{a}^+ - \eta^*\hat{a})], \end{aligned} \tag{9}$$

where $\hat{a}$ and $\hat{a}^+$ are the annihilation and creation operators for the field mode. For the coherent superposition state of density matrix $\rho = |\phi\rangle\langle\phi|$, the corresponding Wigner function can be written as

$$\begin{aligned} W_{\mathrm{sch}}(\alpha) = \frac{2N^2}{\pi}\{&\exp(-2|\alpha-\beta|^2) \\ &+ \exp(-2|\alpha+\beta|^2) + \exp(-2|\alpha|^2) \\ &\cdot\{\exp[2(\alpha\beta^* - \alpha^*\beta) + i\varphi] \\ &+ \exp[-2(\alpha\beta^* - \alpha^*\beta) - i\varphi]\}\}. \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (6), we obtain the expression of the Wigner function of the teleported state,

$$\begin{aligned} &W_{\mathrm{sch}}(\alpha_{\mathrm{out}}) \\ &= \frac{2N^2}{\pi\lambda^2(2\tau+1)}\{\exp[-\frac{2}{2\tau+1}(|\beta|^2 + \frac{|\alpha_{\mathrm{out}}|^2}{\lambda^2})] \\ &\cdot\{\exp[\frac{2}{\lambda(2\tau+1)}(\alpha_{\mathrm{out}}^*\beta + \alpha_{\mathrm{out}}\beta^*)] \\ &+ \exp[-\frac{2}{\lambda(2\tau+1)}(\alpha_{\mathrm{out}}^*\beta + \alpha_{\mathrm{out}}\beta^*)]\} \\ &+ \exp(-\frac{4\tau}{2\tau+1}|\beta|^2 - \frac{2|\alpha_{\mathrm{out}}|^2}{\lambda^2(2\tau+1)}) \\ &\cdot\{\exp[\frac{2}{\lambda(2\tau+1)}(\alpha_{\mathrm{out}}\beta^* - \alpha_{\mathrm{out}}^*\beta) + i\varphi] \\ &+ \exp[-\frac{2}{\lambda(2\tau+1)}(\alpha_{\mathrm{out}}\beta^* - \alpha_{\mathrm{out}}^*\beta) - i\varphi]\}\}. \end{aligned} \tag{11}$$

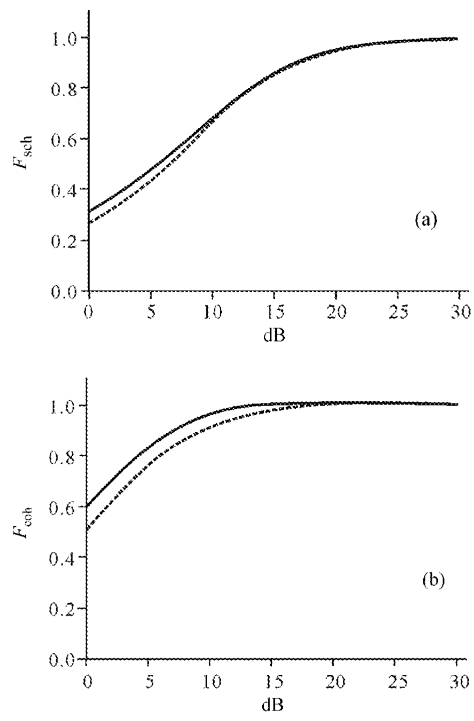

Fig. 1. The solid line denotes the fidelity at the optimal gain λ_{opt}, while the dashed line denotes the fidelity at the gain $\lambda = 1$. (a) For the teleportation of the Schrödinger-cat state $(|\beta\rangle + |-\beta\rangle)$ with $\beta = 1.5i$; (b) for the teleportation of the coherent state $|\beta\rangle$ with $\beta = 1.5i$.

One performance measure for teleportation is the fidelity between the input state and the teleported state. It is an important quantity in describing the transmission of quantum information through quantum channels. For a pure input state, $\hat{\rho}_{\rm in} = |\psi_{\rm in}\rangle\langle\psi_{\rm in}|$, the input-output fidelity is defined as $F = \langle\psi_{\rm in}|\hat{\rho}_{\rm out}|\psi_{\rm in}\rangle = \mathbf{Tr}[\hat{\rho}_{\rm in}\hat{\rho}_{\rm out}]$, where $\hat{\rho}_{\rm out}$ indicates the output density operator. Equivalently, the fidelity can also be represented by the overlap of the Wigner function[15],

$$F = \pi\int {\rm d}^2\alpha W_{\rm in}(\alpha)W_{\rm out}(\alpha). \tag{12}$$

For a maximally entangled channel and $\lambda = 1$, perfect teleportation is achieved and the pure input state is reproduced at the receiving station. Whereas, in the case of a partially entangled quantum channel, the fidelity is always less than unity. The aim of the present paper is to choose a proper displacement parameter λ to maximize the fidelity.

In terms of Eqs. (10) and (11), the teleportation fidelity of the coherent superposition state can be calculated as

$$\begin{aligned}F_{\rm sch} = {} & \frac{4N^4}{\lambda^2(2\tau+1)+1}\{\exp[\frac{-2(1-\lambda)^2}{\lambda^2(2\tau+1)+1}|\beta|^2] \\ & +\exp[\frac{-2(1+\lambda)^2}{\lambda^2(2\tau+1)+1}|\beta|^2] \\ & +2\cos\varphi\{\exp[-\frac{4\lambda^2(\tau+1)}{\lambda^2(2\tau+1)+1}|\beta|^2] \\ & +\exp[-\frac{4(\lambda^2\tau+1)}{\lambda^2(2\tau+1)+1}|\beta|^2]\} \\ & +\cos(2\varphi)\exp[-\frac{2\lambda^2(4\tau+1)+4\lambda+2}{\lambda^2(2\tau+1)+1}|\beta|^2] \\ & +\exp[-\frac{2\lambda^2(4\tau+1)-4\lambda+2}{\lambda^2(2\tau+1)+1}|\beta|^2]\}. \end{aligned} \tag{13}$$

Thus we obtain a general formula of fidelity for teleportation of Schrödinger cat states by using the quantum channel of the two-mode squeezed vacuum. Note that if the displacement gain $\lambda = 1$, the fidelity $F_{\rm sch}$ is the same as the one described by Eq. (12) in Ref. [3]. For $\lambda = 1$ and $r \to \infty$, as we expect, the fidelity of the teleportation will approach unity. Moreover, when the displacement gain $\lambda = 1$ and zero squeezing $r = 0$, we have

$$\begin{aligned}F_{\rm sch} &= N^4[1+2\exp(-2|\beta|^2)+4\cos\varphi\exp(-2|\beta|^2) \\ &\quad +\cos 2\varphi\exp(-4|\beta|^2)] \\ &= \frac{1}{2} - N^4[1-\exp(-2|\beta|^2)]^2. \end{aligned} \tag{14}$$

It implies that the fidelity for zero squeezing of the initial entanglement resource is much dependent on the coherent superposition input state, and the fidelity satisfies $\frac{1}{4} \le F_{\rm sch} \le \frac{1}{2}$.

Next we consider the teleportation of a coherent input state $|\psi\rangle = |\beta\rangle$, which has a Poisson photon number distribution with a mean of $\bar{n} = |\beta|^2$. The Wigner function of the state can be given by

$$W_{\rm coh}(\alpha) = \frac{2}{\pi}\exp(-2|\alpha-\beta|^2). \tag{15}$$

With the same technique above, we can carry out the teleported Wigner function of the coherent state

$$\begin{aligned}W_{\rm coh}(\alpha_{\rm out}) = {} & \frac{2}{\pi\lambda^2(2\tau+1)} \\ & \cdot\exp[-\frac{2}{2\tau+1}|\beta|^2 - \frac{2}{\lambda^2(2\tau+1)}|\alpha_{\rm out}|^2 \\ & +\frac{2}{\lambda(2\tau+1)}(\alpha_{\rm out}^*\beta+\alpha_{\rm out}\beta^*)]. \end{aligned} \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (12), the fidelity of teleportation of the coherent state can be described by

$$\begin{aligned}F_{\rm coh} = {} & \frac{2}{\lambda^2(2\tau+1)+1} \\ & \cdot\exp[-(1-\lambda)^2\frac{2|\beta|^2}{\lambda^2(2\tau+1)+1}]. \end{aligned} \tag{17}$$

In the case of $\lambda = 1$ and the squeezing parameter $r = 0$, we get $F_{\rm coh} = \frac{1}{2}$, which is the classical fidelity limit for teleporting any coherent states[21,22]. It should be noted the difference between the two cases: for the teleportation of the coherent states, the fidelity $F_{\rm coh} = \frac{1}{1+\exp(-2r)}$ at $\lambda = 1$ is independent of the input states, while for the teleportation of the Schrödinger cats, the fidelity $F_{\rm sch}$ at $\lambda = 1$ is dependent on the input states.

Finally, we perform numerical computation about the teleportation fidelity as well as the input and teleported Wigner function. Figure 1 shows a comparison between the fidelity for the unity gain and for the optimal gain $\lambda_{\rm opt}$ respectively, which is chosen to maximize the corresponding fidelity. We can see that in the case of small squeezing, the optimal fidelity is larger than the non-optimized fidelity with the local displacement $\lambda = 1$ for teleporting both Schrödinger cats and coherent states. That is to say, when the entanglement of the initial squeezed state is small, the optimal value of the displacement gain $\lambda_{\rm opt}$ is not equal to one for the teleportation of the two specific input states. However, in the case of large squeezing, the fidelity corresponding to $\lambda = 1$ will become the optimal fidelity, which is also true for teleporting any other input states. It can be explained by means of the Heisenberg representation[8,22]: in our protocol, $\hat{x}_2 = \hat{x}_{\rm in} + (\hat{x}_1 + \hat{x}_2) - \sqrt{2}\hat{x}_a$ and $\hat{p}_2 = \hat{p}_{\rm in} - (\hat{p}_1 - \hat{p}_2) - \sqrt{2}\hat{p}_b$, where $\hat{x}_1 + \hat{x}_2 = 2{\rm e}^{-r}\hat{x}_1^{(0)}$ and $\hat{p}_1 - \hat{p}_2 = 2{\rm e}^{-r}\hat{p}_2^{(0)}$ with a superscript denoting vacuum modes. The output state in mode 2 of the entanglement resource is then expressed as $\hat{x}_{\rm out} = \hat{x}_2 + \lambda\sqrt{2}x_a$ and $\hat{p}_{\rm out} = \hat{p}_2 + \lambda\sqrt{2}p_b$. For the high entanglement of the initial two-mode squeezed state, $\hat{x}_1 + \hat{x}_2 \to 0$ and $\hat{p}_1 - \hat{p}_2 \to 0$, the ideal result $\hat{\alpha}_{\rm out} = \hat{x}_{\rm out} + i\hat{p}_{\rm out} \to \hat{\alpha}_{\rm in}$ can only be achieved when the gain $\lambda = 1$ is satisfied. It shows that for large squeezing of the entanglement resource, the displacement gain $\lambda = 1$ is the best choice for the teleportation of arbitrary input states. Therefore, from Fig. 1, we also learn that the method of the fidelity optimization is suitable for the case of small squeezing. The Wigner function of a Schrödinger-cat state and of a coherent state are plotted in Figs. 2 and 3, respectively. Clearly, Fig. 2(c) is similar to Fig. 2(b), since there is only a little difference between the corresponding fidelities: the former $F_{\rm sch} = 0.6773$, the latter $F_{\rm sch} = 0.6602$. As shown in Fig. 3, the output state for the optimal

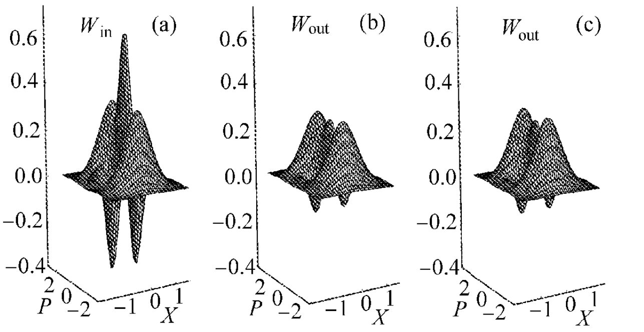

Fig. 2. (a) The Wigner function of the Schrödinger cat state $(|\beta\rangle + |-\beta\rangle)$ with $\beta = 1.5i$; the Wigner function of the output state with the squeezing parameter $r = 1.15$ (10 dB), (b) for the gain $\lambda = 1$ and (c) for the optimal gain $\lambda_{\rm opt} = 0.9492$, respectively.

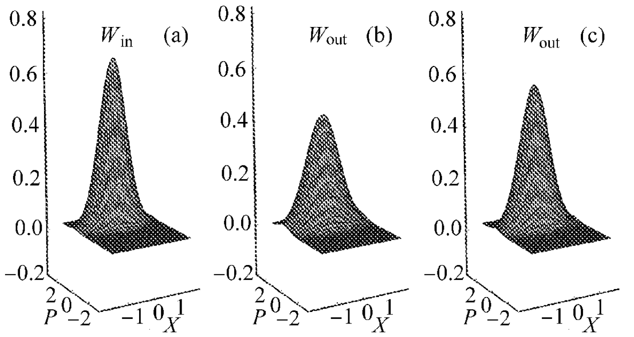

Fig. 3. (a) The Wigner function of the coherent state $|\beta\rangle$ with $\beta = 1.5i$; the Wigner function of the output state with the squeezing parameter $r = 0.69$ (6 dB), (b) for the gain $\lambda = 1$ and (c) for the optimal gain $\lambda_{\rm opt} = 0.8408$, respectively.

displacement (c) is more similar to the input state (a) than the output state for unity gain (b). Hence, in virtue of selecting the optimal gain, the fidelity of the teleportation of the coherent state is improved clearly. It means that to some extent the displacement gain has important influence on the teleportation fidelity. The point is that the gain is an extra degree of freedom, we can optimize the performance of fidelity by tuning it. Within the reach of current technology, the displacement operation can be efficiently performed using a beam splitter of a high transmittance[4,23].

In summary, the fidelity of continuous-variable quantum teleportation can be optimized by means of changing the displacement of local operation in the final reconstruction process of the output field. The point is that the displacement gain is an extra parameter, which allows a maximization of the fidelity. The specific results for the teleportation of Schrödinger cats, and of coherent states have been achieved. These results show that for small squeezing, the local displacement corresponding to the maximum fidelity is not equal to one. In the case of large squeezing, unity displacement gain is the best choice for the teleportation of arbitrary quantum states. Hence, the optimization of the gain dependence of the fidelity is suitable for small squeezing of the entanglement resource.

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