

Solutions of the modification heating conduction equations of a kind of laser thermal effect

Lingyun Zhou (周凌云)¹, Canbang Zhang (张灿邦)^{1,2}, Yiyi Zhou³, and Guangmin Wu (吴光敏)¹

¹Reseach Center of Nonlinear Science, Kunming University of Science and Technology, Kunming 650093

²Department of Physics, Honghe College, Mengzi 661100

³Department of Physics and Astronomy, Georgia State University, Atlanta, Georgia 30303-3083, USA

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This paper has solved the Chester modified heat conduction equation of the different relaxation time τ value under different temperature conditions, different boundary conditions and the different initial conditions by different means of methods. These solutions can help to obtain temperature field of laser thermal effects.

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In Ref. [1], the modified heat conduction Eq. (1) has been obtained from the microscopic points of view. The equation is

$$\frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau} \frac{\partial T}{\partial t} - \frac{K}{\tau c} \nabla^2 T = 0, \quad (1)$$

where τ is a relaxation time, c is the heat capacity per unit volume, K is the thermal conductivity (the Fourier equation is $\frac{c}{K} \frac{\partial T}{\partial t} - \nabla^2 T = 0$) and $\frac{k}{\tau c} = v_0^2$, v_0 is the limiting velocity of the propagation of a thermal signal^[2]. It is difficulty to solve the Eq. (1). In order to solve Eq. (1), we must find out value of τ . According to quantum theory of solid, the relaxation time τ is associated with the communication "time" between phonons (phonon-phonon collisions) for the commencement of resistive flow^[3]. It determines the "critical frequency", the attenuation of "thermal wave" and temperature T ^[2,3]. And when temperature T is higher, there is $\tau = (BT^3\omega^2)^{-1}$ for the mechanism of heat transference is phonon-phonon scatter^[3], under general temperature, $\tau = A\omega$.

First of all we solve Eq. (1) for $\tau = A\omega^4$, where A is a constant that relates to matter structure and ω is lattice vibration frequency. Under general temperature, ω can be seen as constant. Hence τ is a constant.

Now we solve Eq. (1) by travelling wave method and taking only x -coordinate. Let $\xi = x - vt$, $T(x, t) = T(\xi)$, Eq. (1) becomes $(v^2 - \frac{k}{\tau c}) \frac{d^2 T}{d\xi^2} - \frac{v}{c} \frac{dT}{d\xi} = 0$. After simple integral calculation, we can obtain the travelling wave solution of the above equation $T(x, t) = D \exp \left[\frac{v}{\tau} \cdot \frac{x-vt}{v^2-v_0^2} \right]$, where D is integral constant, $v_0 > v$. The travelling wave solution is no sense for that sets temperature field of laser thermal effect. So we must obtain other solutions of Eq. (1), as following.

The modified heat conduction Eq. (1) for one-dimension unbounded space of instantaneous point heat source can be write as

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{v_0^2} \frac{\partial^2 T}{\partial t^2} - \frac{1}{v_0^2 \tau} \frac{\partial T}{\partial t} = -\delta(x-x')\delta(t-t'),$$

$$(t, t' > 0, -\infty < x, x' < \infty), \quad (2)$$

$$T|_{t=0} = 0, \left. \frac{\partial T}{\partial t} \right|_{t=0} = 0, \quad (-\infty < x, x' < \infty), \quad (3)$$

where v_0 and τ are constants. To solve Eqs. (2) and (3), let $T = e^{\alpha t} v$, $T_{xx} = e^{\alpha t} v_{xx}$, and α awaits determine. It can be obtained that

$$T_{xx} - \frac{1}{\tau} T_t - \frac{1}{a^2} T_{tt}$$

$$= e^{\alpha t} \left[v_{xx} - \frac{1}{a^2} v_{tt} - \left(\frac{1}{k} + \frac{2\alpha}{a^2} \right) v_t - \left(\frac{\alpha}{k} + \frac{\alpha^2}{a^2} \right) v \right]$$

$$= -\delta(x-x')\delta(t-t'). \quad (4)$$

Let $\frac{1}{k} + \frac{2\alpha}{a^2} = 0$, $\alpha = -\frac{a^2}{2k}$, v satisfies

$$\begin{cases} v_{xx} - \frac{1}{a} v_{tt} + \frac{a^2}{4k^2} v = -e^{-\alpha t} \delta(x-x')\delta(t-t') \\ = -e^{\frac{a^2}{2k} t} \delta(x-x')\delta(t-t'), \\ -\infty < x < \infty, t > t' \\ v|_{t=0} = 0, v_t|_{t=0} = 0 \end{cases} \quad (5)$$

Let $\Phi(x, t, z) = e^{\frac{a}{2k} z} v$, it can be obtained that

$$\begin{cases} \Phi_{xx} + \Phi_{zz} - \frac{1}{a^2} \Phi_{tt} = \left[v_{xx} - \frac{1}{a} v_{tt} + \left(\frac{a}{2k} \right)^2 v \right] e^{\frac{a}{2k} z} \\ = -e^{\frac{a^2}{2k} t + \frac{a}{2k} z} \delta(x-x')\delta(t-t') \\ \Phi|_{t=0} = 0, \Phi_t|_{t=0} = 0 \end{cases} \quad (6)$$

Let

$$\Phi = \int_0^t \psi(z, z, \tau_1) d\tau_1, \quad (7)$$

ψ satisfies

$$\begin{cases} \psi_{xx} + \psi_{zz} - \frac{1}{a^2} \psi_{tt} = 0 \\ \psi|_{t=\tau_1} = 0, \\ \psi_t|_{t=\tau_1} = e^{\frac{a^2}{2k} \tau_1 + \frac{a}{2k} z} \alpha^2 \delta(x-x')\delta(\tau_1 - t'). \end{cases} \quad (8)$$

Let $t - \tau_1 = y$, Eq. (8) becomes

$$\begin{cases} \psi_{xx} + \psi_{zz} - \frac{1}{a^2} \psi_{yy} = 0 \\ \psi|_{y=0} = 0, \psi_y|_{y=0} = e^{\frac{a^2}{2k} \tau_1 + \frac{a}{2k} z} \delta(x-x')\delta(\tau - t'). \end{cases} \quad (9)$$

We can obtain solution

$$\psi(x, z, y) = \frac{a^2}{2\pi a} \int_{D_{ay}^M} \frac{e^{\frac{a^2}{2k}\tau_1 + \frac{a}{2k}z} \delta(\xi - x') \delta(\tau_1 - t')}{\sqrt{(ay)^2 - (\xi - x)^2 - (y - z)^2}} d\xi dy, \quad (10)$$

where $D_{ay}^M : (\xi - x)^2 + (y - z)^2 \leq (ay)^2$ (see Fig. 1), and

$$\psi(x, y, z) = \frac{a^2 \delta(\tau_1 - t') e^{\frac{a^2}{2k}(\tau + z)}}{2\pi a} \int_{-1}^{+1} \frac{e^{i\frac{a}{2k}y\sqrt{(x'-x)^2 - a^2(t-t')^2}}}{\sqrt{1 - y^2}} dy. \quad (11)$$

From Eqs. (7) and (11) we can obtain

$$\Phi = \frac{ae^{\frac{a^2}{2k}(t'+z)}}{2\pi} \int_{-1}^{+1} \frac{e^{-i\frac{a}{2k}y\sqrt{(x'-x)^2 - a^2(t-t')^2}}}{\sqrt{1 - y^2}} dy,$$

and then

$$v = \frac{ae^{\frac{a^2}{2k}t}}{2\pi} \int_{-1}^{+1} \frac{e^{-i\frac{a}{2k}y\sqrt{(x'-x)^2 - a^2(t-t')^2}}}{\sqrt{1 - y^2}} dy,$$

$$T = \frac{ae^{-\frac{a^2}{2k}(t-t')}}{2\pi} \int_{-1}^{+1} \frac{e^{i\frac{a}{2k}y\sqrt{(x'-x)^2 - a^2(t-t')^2}}}{\sqrt{1 - y^2}} dy.$$

The Bessel function is $J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\sqrt{\pi}(\nu + \frac{1}{2})} \int_{-1}^{+1} e^{izt} (1 - t^2)^{\nu - \frac{1}{2}} dt$. Hence $\int_{-1}^{+1} \frac{e^{-i\frac{a}{2k}y\sqrt{(x'-x)^2 - a^2(t-t')^2}}}{\sqrt{1 - y^2}} dy = \pi J_0 \left[\frac{a}{2k} \sqrt{(x - x')^2 - v_0^2(t - t')^2} \right]$.

We can obtain the solution of Eqs. (2) and (3) as

$$T = \frac{v_0}{2} e^{-\frac{1}{2\tau}(t-t')} J_0 \left[\frac{1}{2v_0\tau} \sqrt{(x - x')^2 - v_0^2(t - t')^2} \right], \quad (12)$$

where J_0 is 0-order Bessel function. When $v_0 \rightarrow \infty$, $\tau \rightarrow$, Eq. (2) degenerates to corresponding Fourier heat conduction equations, and the solution Eq. (12) degenerates to the solution of the corresponding Fourier heat conduction equation

$$T = \sqrt{\frac{c}{4k(t-t')}} \exp \left[\frac{k(x-x')^2}{4c(t-t')} \right]. \quad (13)$$

So far, the temperature field distribution that heated by pulse laser utilizes the formal Eq. (13) still, but the formal Eq. (13) is not reasonable^[4]. H. Salzmann points out that Fourier heat conduction theory is not reasonable for heat conduction effect by ultrashort pulse laser to heat material^[5]. Hence we must use modified heat conduction theory.

A special example on the analogy of Eqs. (2) and (3)

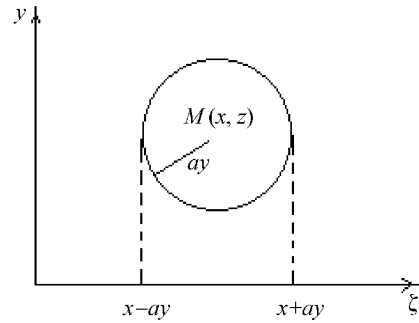


Fig. 1. Region of D_{ay}^M .

is the modification heat conduction of infinite slab. Its solution is

$$T - T_r = (T_m - T_r) \exp \left(-\frac{\xi}{\sqrt{2}} x \right) \cos \left(\omega_T t - \frac{\eta}{\sqrt{2}} x \right), \quad (14)$$

where T_r is a reference temperature, T_m is maximum temperature (obviously, T_m relates to laser action time and medium behaviour), ω_T is the frequency of temperature variation, and

$$\xi = \left\{ \left[\frac{\omega_T^4}{v_0^4} + \left(\frac{\omega_T \rho c}{k} \right)^2 \right]^{1/2} - \frac{\omega_T^2}{v_0^2} \right\}^{1/2},$$

$$\eta = \left\{ \left[\frac{\omega_T^4}{v_0^4} + \left(\frac{\omega_T \rho c}{k} \right)^2 \right]^{1/2} + \frac{\omega_T^2}{v_0^2} \right\}^{1/2}.$$

The solution (14) represents a temperature wave moving into the slab with the velocity $v = \sqrt{2}\omega_T/\eta$. Since all the properties of the material have been assumed to be independent of temperature, then for a slab of given material, v is a function of ω_T only. It therefore assumes its minimum value $v = 0$ when $\omega = 0$, which corresponds to no temperature change at the surface and its maximum value $v = v_0$ when $\omega = \infty$, which corresponds to an impulse.

When immediately introduce laser thermal effect, the Eq. (1) and its boundary conditions and initial conditions are

$$T_{tt} + \frac{1}{\tau} T_t - v_0^2 T_{xx} = 0, \quad \tau = a\omega^4, \quad v_0^2 = \frac{k}{\tau c}, \quad (15)$$

$$T(x, t)|_{t=0} = T_0, \quad \frac{\partial T}{\partial t} \Big|_{t=0} = 0, \quad (16)$$

$$-k \left(\frac{\partial T}{\partial x} \right) \Big|_{x=0} = a_A p(t), \quad (\text{see Ref. [4]}) \quad (17)$$

$$\lim_{x \rightarrow +\infty} T(x, t) = T_0, \quad (18)$$

where a_A is absorptance of material and $p(t)$ is laser power. Let $b = a_A p(t)$, $a_A p(t)$ can be regarded as a constant. We solve the Eq. (15) (include its boundary

conditions and initial conditions) by means of Laplace transform^[8]. Let

$$U(x, p) = \mathcal{L}[T(x, t)] = \int_0^\infty T(x, t)e^{-pt} dt, \quad (19)$$

taking Laplace transformation for Eq. (15) and using conditions (16) and (17), we can obtain $p^2 U(x, p) - pT(x, 0) - (\frac{\partial T}{\partial t})_{t=0} + \frac{1}{\tau} pU(x, p) - \frac{1}{\tau} T(x, 0) - u_0^2 U_{xx}(x, p) = 0$.

After straighten out, it can be obtained that

$$\frac{d^2 U(x, p)}{dx^2} - \frac{1}{v_0^2} \left(p^2 + \frac{p}{\tau} \right) U(x, p) = -\frac{T_0}{v_0^2} \left(p + \frac{1}{\tau} \right). \quad (20)$$

The corresponding homogeneous Eq. (20) is

$$U''_{xx}(x, p) - \frac{1}{v_0^2} \left(p^2 + \frac{p}{\tau} \right) U(x, p) = 0. \quad (21)$$

The characteristic Eq. (21) is

$$r^2 - \frac{1}{v_0^2} \left(p^2 + \frac{p}{a} \right) = 0. \quad (22)$$

The characteristic root of Eq. (22) is

$$r_{1,2} = \pm \frac{1}{v_0} \sqrt{p^2 + \frac{p}{a}}, \quad (v_0 > 0). \quad (23)$$

The general solution of Eq. (21) is

$$\tilde{U}(x, p) = C_1 e^{\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}}} + C_2 e^{-\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}}}.$$

Taking the particular solution of Eq. (20)

$$U^*(x, p) = F(p)(Gx^2 + Hx + K)$$

to substitute to Eq. (20), we can obtain

$$U^*(x, p) = \frac{T_0}{p} \quad (\text{or take } U^*(x, p) = GF(p)).$$

Hence, the general solution of Eq. (20) is

$$U(x, p) = \tilde{U} + U^* = C_1 e^{\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}}} + C_2 e^{-\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}}} + \frac{T_0}{p}. \quad (24)$$

Taking Laplace transformation for Eqs. (17) and (18), we can obtain

$$U'_x(0, p) = -\frac{b}{\lambda p}, \quad (25)$$

$$\lim_{x \rightarrow +\infty} U(x, p) = \frac{T_0}{p}, \quad (26)$$

From Eq. (26), we can obtain $C_1 = 0$. From Eq. (25), can obtain $C_2 = \frac{bv_0}{\lambda p \sqrt{p^2 + \frac{p}{a}}}$. Substituting into Eq. (24), we can obtain the solution of Eq. (20)

$$U(x, p) = \frac{T_0}{p} + \frac{bv_0}{\lambda p \sqrt{p^2 + \frac{p}{a}}} e^{-\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}}}, \quad (27)$$

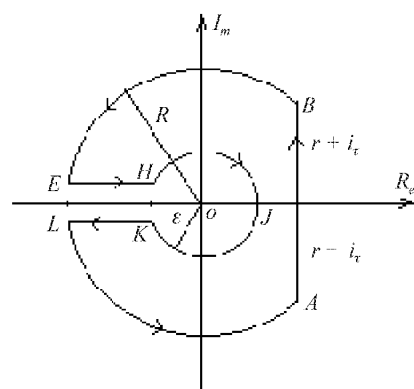


Fig. 2. Integral contour of I .

where $b = a_{AP}$, $a = \tau = A\omega^4$. From Eq. (27), it can be obtained that

$$T(x, t) = \mathcal{L}^{-1}\{U(x, p)\} = T_0 + \frac{bv_0}{2\pi i \lambda} \int_{r-i\infty}^{r+i\infty} \frac{e^{-\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}} + pt}}{p \sqrt{p^2 + \frac{p}{a}}} dp. \quad (28)$$

Let $I = \frac{bv_0}{2\pi i \lambda} \int_{r-i\infty}^{r+i\infty} \frac{e^{-\frac{\sigma}{v_0} \sqrt{p^2 + \frac{p}{a}} + pt}}{p \sqrt{p^2 + \frac{p}{a}}} dp$, we can obtain $I = \lim_{R \rightarrow \infty, \epsilon \rightarrow 0} \frac{bv_0}{2\pi i \lambda} [\int_{EH} + \int_{HJK} + \int_{KL}]$.

Lastly, we obtain the solution of Eq. (25) and conditions (16) and (18)

$$T(x, t) = T_0 + \frac{v_0 a_{AP}(t)}{\pi \lambda} \int_0^{+\infty} \frac{e^{-vt} \cos\left(\frac{\sqrt{\frac{v}{A\omega^4} - v^2}}{v_0} x\right)}{v \sqrt{\frac{v}{A\omega^4} - v^2}} dv. \quad (29)$$

From the solution of Eq. (29), we can obtain the temperature field distribution of a kind of laser thermal effect.

When temperature is high, according to solid quantum theory, the relaxation time^[3] is $\tau = (BT^3\omega^2)^{-1}$, then Eq. (1) can be written as

$$T_{tt} + BT^3\omega^2 T_t - v_0^2 T_{xx} = 0, \quad (30)$$

where B is undetermined coefficient which relates to the mechanism of the phonon-phonon scatter in the material. We solve Eq. (30) by travelling wave method. Let $\zeta = x - vt$, Eq. (30) becomes

$$(v_0^2 - v^2)T_{\zeta\zeta} + BT^3\omega^2 v T_{\zeta} = 0. \quad (31)$$

Let $R^2 = v_0^2 - v^2 > 0$, $B\omega^2 v = L$, and integrating Eq. (31), we obtain

$$\left(\frac{1}{4}LT^4 - C_1\right)^{-1} dT = -R^{-2} d\zeta, \quad (32)$$

where C_1 is integration constant. If $C_1 > 0$, we can obtain travelling wave solution of Eq. (30)

$$-\frac{1}{R^2 C_1} \sqrt[4]{\frac{4C_1}{L}} (x - vt) = \frac{1}{4} \left[\ln \left(C_2 \frac{\sqrt[4]{\frac{L}{2C_1}} T - 1}{\sqrt[4]{\frac{L}{2C_1}} T + 1} \right) - \frac{1}{2} \arctan \left(\sqrt[4]{\frac{L}{4C_1}} T \right) \right], \quad (33)$$

where C_2 is integration. If $C_1 < 0$, we can obtain travelling wave solution of Eq. (30)

$$-\frac{4C_1}{R^2} (x - vt) = \sqrt[4]{\frac{L}{-4C_1}} \left[\frac{1}{4\sqrt{2}} \ln \left(C_3 \frac{y^2 + y\sqrt{2} + 1}{y^2 + y\sqrt{2} - 1} \right) + \frac{1}{2\sqrt{2}} \arctan \left(\frac{y\sqrt{2}}{1 - y^2} \right) \right], \quad (34)$$

where $y = \sqrt[4]{\frac{L}{-4C_1}} T$, C_3 is integration. The solutions of

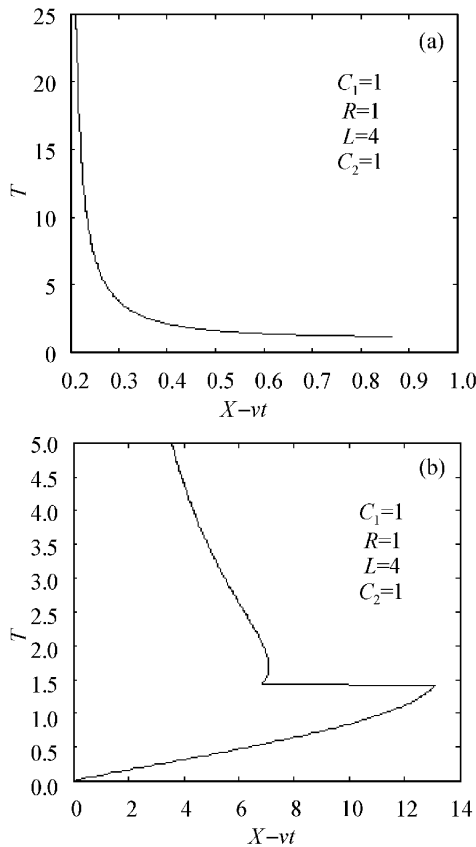


Fig. 3. (a) The solution with $C_1 > 0$; (b) the solution with $C_1 < 0$.

Eqs. (33) and (34) show that the nonlinear heat conduction Eq. (30) has some integrability. Through numerical calculation and according to Eqs. (33) and (34), we can obtain schematic diagrams of the $\zeta - T$ function (as shown in Figs. 3(a) and (b)). Obviously, Fig. 3(b) is not reasonable, hence C_1 can be only greater than zero ($C_1 > 0$).

The modified heat conduction Eq. (1) was derived in companion piece of this paper^[1]. It is very difficult that to solve Eq. (1), moreover, the solution of Eq. (1) is dependent on τ . The relaxation time τ is the crucial parameter^[2]. According to the quantum theory of solid^[3], the relaxation time τ is related to “thermal wave” and temperature. When τ is independent of temperature T (i.e. $\tau = A\omega^4$ can be see a constant), the corresponding Eq. (1) is a linear differential equation. We solved the Eq. (1) of $\tau = \text{constant}$ under different conditions by means of different methods and obtained analytic solutions. These solutions can help to obtain temperature field of thermal effect of laser^[9]. When τ is dependent on temperature T , the Eq. (1) changes into a nonlinear differential equation (i.e. Eq. (30)). It is very difficult and complex that to solve the nonlinear heat conduction Eq. (30). As space of the paper is limited, we solve Eq. (30) by travelling wave method only. We will discuss minutely the solution of Eq. (30) by different methods in other paper.

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