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n 阶波相互作用理论

谭维翰

（中国科学院上海光机所）

提要： 文献[1]已研究了光与二能级原子相互作用孤立波方程的准确解。本文推广[1]的讨论，先求出包括二波、三波在内的 n 阶波相互作用方程，然后用逆散射方法求其解。

On the nth order wave interaction theory

Tan Weihan

(Shangahai Institute of Optics and Fine Machanics, Academia Sinica)

Abstcract: The nth order wave interaction theory investigated in this paper should be considered as a continuation of the "exact solution to solitary wave equations of light interaction with two-level atomic systems" in the earlier paper[1]. In literature[4,5] the third order wave equations originate from the research of three-wave interaction. But the nth order wave equations considered now are derived from the n-dimensionall vector functions which satisfy two nonlinear equations, and the general solutions can be expressed in terms of the upper hemiplane analytical solutions and the lower hemiplane analytical solutions. These results include the 2nd and the 3rd order solutions as a special example.

一、引　　言

用逆散射方法解非线性波相互作用是近年来研究得较多的一课题[1~7]。亦即不直接解非线波相互作用方程 $\phi_t=K(\phi)$，而是找出依赖于 ϕ 的线性算子 L、B，使得满足如下方程，并用‘逆方法’求解

$$i\psi_t=B\psi \tag{1}$$

$$L\psi=E\psi \tag{2}$$

$$iL_t=BL-LB \tag{3}$$

方程(1)是关于 t 的一阶偏微分方程，容易推广到 n 阶波相互作用，而 L 则是关于 x 的二阶偏微分方程，要找出又满足(2)、(3)的 L、B 是不容易的。本文不用(2)、(3)，而是用一形状与(1)相似的关于 x 的一阶偏微分方程来代替。文献[7]虽提到交叉求导方式，但未采用矩阵形式。也未涉及多于二阶波相互作用方程。

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二、n 阶波相互作用方程

下面设 v 为 n 维空间的矢函数，Q、B 为 n 阶函数矩阵，而 v，B，Q 满足如下微分方程。

$$\frac{\partial v}{\partial x}=Qv,\ \frac{\partial v}{\partial t}=Bv \tag{4}$$

又设在 $x-t$ 平面上，(4)式有 n 个线性无关的矢函数解。则由(4)得

$$\frac{\partial}{\partial t}(Qv)=\frac{\partial}{\partial x}(Bv)$$

$$\left(\frac{\partial Q}{\partial t}-\frac{\partial B}{\partial x}\right)v=B\frac{\partial v}{\partial x}-Q\frac{\partial v}{\partial t}$$

$$=(BQ-QB)v$$

即

$$\frac{\partial Q}{\partial t}-\frac{\partial B}{\partial x}=BQ-QB \tag{5}$$

(4)与(5)即 n 阶波方程。由(5)可得出二、三阶波方程及原子与辐射作用方程。

① $n=2$

$$\left.\begin{aligned}&v=\begin{pmatrix}v_1\\v_2\end{pmatrix},\ Q=\begin{pmatrix}-i\zeta & q\\ r & i\zeta\end{pmatrix},\\ &B=\begin{pmatrix}-ia & -ib\\ -ic & ia\end{pmatrix}\end{aligned}\right\} \tag{6}$$

代入(5)式便得二阶波相互作用方程的关系[3]

$$\begin{aligned}&a_x=cq-br\\ &b_x+2i\zeta b=iq_t-2aq\\ &c_x-2i\zeta c=ir_t+2ar\end{aligned} \tag{7}$$

② $n=3$

$$v=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix},\ Q=\begin{pmatrix}Q_{11} & Q_{12} & Q_{13}\\ Q_{21} & Q_{22} & Q_{23}\\ Q_{31} & Q_{32} & Q_{33}\end{pmatrix}$$

$$B=\begin{pmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{pmatrix}$$

现将 Q_{ij}，B_{ij} 取为如下形状，令 Q_{ii}，B_{ii} 为常数。

$$\left.\begin{aligned}&Q=\begin{pmatrix}Q_{11} & u_3 & -u_2^*\\ -u_3^* & Q_{22} & u_1\\ u_2 & -u_1^* & Q_{33}\end{pmatrix},\\ &B=\begin{pmatrix}B_{11} & c_3u_3 & -c_2u_2^*\\ -c_3u_3^* & B_{22} & c_1u_1\\ c_2u_2 & -c_1u_1^* & B_{33}\end{pmatrix}\end{aligned}\right\} \tag{8}$$

又令

$$\left.\begin{aligned}&c_1=\frac{B_{22}-B_{33}}{Q_{22}-Q_{33}},\quad c_2=\frac{B_{33}-B_{11}}{Q_{33}-Q_{11}},\\ &c_3=\frac{B_{11}-B_{22}}{Q_{11}-Q_{22}}\end{aligned}\right\} \tag{9}$$

将(8)代入(5)，并注意到(9)，便得

$$\begin{pmatrix}0 & \frac{\partial u_3}{\partial t}-c_3\frac{\partial u_3}{\partial x} & -\frac{\partial u_2^*}{\partial t}+c_2\frac{\partial u_2^*}{\partial x}\\ -\frac{\partial u_3^*}{\partial t}+c_3\frac{\partial u_3^*}{\partial x} & 0 & \frac{\partial u_1}{\partial t}-c_1\frac{\partial u_1}{\partial x}\\ \frac{\partial u_2}{\partial t}-c_2\frac{\partial u_2}{\partial x} & -\frac{\partial u_1^*}{\partial t}+c_1\frac{\partial u_1^*}{\partial x} & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & -(c_1-c_2)u_1^*u_2^* & -(c_1-c_3)u_1u_3\\ -(c_2-c_1)u_1u_2 & 0 & -(c_2-c_3)u_2^*u_3^*\\ -(c_3-c_1)u_1^*u_3^* & -(c_3-c_2)u_2u_3 & 0\end{pmatrix}$$

即

$$\begin{aligned}&u_{1t}-c_1u_{1x}=(c_3-c_2)u_2^*u_3^*\\ &u_{2t}-c_2u_{2x}=(c_1-c_3)u_3^*u_1^*\\ &u_{3t}-c_3u_{3x}=(c_2-c_1)u_1^*u_2^*\end{aligned} \tag{10}$$

易于证明由(10)式表示的三波相互作用满足能量守恒关系

$$\left(\frac{\partial}{\partial t}-c_1\frac{\partial}{\partial x}\right)u_1^2+\left(\frac{\partial}{\partial t}-c_2\frac{\partial}{\partial x}\right)u_2^2+\left(\frac{\partial}{\partial t}-c_3\frac{\partial}{\partial x}\right)u_3^2=0 \tag{11}$$

还可将 (10) 式化为文献[5]中采用过的两种

形式，分别称为爆炸不稳与衰变不稳，令 $u_i=\lambda_i\tilde{u}_i$，(10)式为

$$\begin{aligned}&\tilde{u}_{1t}-c_1\tilde{u}_{1x}=\pm p\tilde{u}_2^*\tilde{u}_3^*\\&\tilde{u}_{2t}-c_2\tilde{u}_{2x}=p\tilde{u}_1^*\tilde{u}_3^*\\&\tilde{u}_{3t}-c_3\tilde{u}_{3x}=p\tilde{u}_1^*\tilde{u}_2^*\end{aligned}\tag{12}$$

式中 p 为

$$\begin{aligned}p&=\frac{(c_2-c_1)\lambda_1\lambda_2}{\lambda_3}\\&=\frac{(c_1-c_3)\lambda_1\lambda_3}{\lambda_2}\\&=\frac{\pm(c_3-c_2)\lambda_3\lambda_2}{\lambda_1}\end{aligned}\tag{13}$$

$$p^3/\lambda_1\lambda_2\lambda_3=\pm(c_3-c_2)(c_2-c_1)(c_1-c_3)\tag{14}$$

$$\begin{aligned}\lambda_1&=\frac{p}{\sqrt{(c_1-c_3)(c_2-c_1)}}\\\lambda_2&=\frac{p}{\sqrt{\pm(c_3-c_2)(c_2-c_1)}}\\\lambda_3&=\frac{p}{\sqrt{\pm(c_3-c_2)(c_1-c_3)}}\end{aligned}\tag{15}$$

③ 若取 Q, B 为

$$Q=\begin{pmatrix}0&Q_3&-Q_2\\-Q_3&0&Q_1\\Q_2&-Q_1&0\end{pmatrix}$$

$$B=\begin{pmatrix}0&B_3&-B_2\\-B_3&0&B_1\\B_2&-B_1&0\end{pmatrix}$$

代入(5)式得

$$\begin{pmatrix}0&\dfrac{\partial Q_3}{\partial t}-\dfrac{\partial B_3}{\partial x}&-\dfrac{\partial Q_2}{\partial t}+\dfrac{\partial B_2}{\partial x}\\-\dfrac{\partial Q_3}{\partial t}+\dfrac{\partial B_3}{\partial x}&0&\dfrac{\partial Q_1}{\partial t}-\dfrac{\partial B_1}{\partial x}\\\dfrac{\partial Q_2}{\partial t}-\dfrac{\partial B_2}{\partial x}&-\dfrac{\partial Q_1}{\partial t}+\dfrac{\partial B_1}{\partial x}&0\end{pmatrix}=\begin{pmatrix}0&B_2Q_1-Q_2B_1&B_3Q_1-Q_3B_1\\B_1Q_2-Q_1B_2&0&B_3Q_2-Q_3B_2\\B_1Q_3-Q_1B_3&B_2Q_3-Q_2B_3&0\end{pmatrix}\tag{16}$$

若 B_i, Q_i 可对易，则(16)式可表示为向量函数形式式。令 $\boldsymbol{Q}=(Q_1,\ Q_2,\ Q_3)$，$\boldsymbol{B}=(B_1,\ B_2,\ B_3)$，则由(16)式得

$$\frac{\partial\boldsymbol{Q}}{\partial t}-\frac{\partial\boldsymbol{B}}{\partial x}=-\boldsymbol{B}\times\boldsymbol{Q}\tag{17}$$

当 B 与 x 无关时，又有

$$\frac{\partial\boldsymbol{Q}}{\partial t}=-\boldsymbol{B}\times\boldsymbol{Q}\tag{18}$$

这就是二能级原子系统与辐射相互作用的矢量表示[8]。

三、n 阶波相互作用方程的解

1．现将波方程(4)、(5)就三阶情形求解，然后推广到 n 阶。对于三阶波方程情形，$\dfrac{\partial v}{\partial x}=Qv$ 可写为

$$\left[-iI\frac{\partial}{\partial x}+i\begin{pmatrix}&Q_{12}&Q_{13}\\Q_{21}&&Q_{23}\\Q_{31}&Q_{32}&\end{pmatrix}+i\begin{pmatrix}Q_{11}&&\\&Q_{22}&\\&&Q_{33}\end{pmatrix}\right]v=0\tag{19}$$

令

$$V=\begin{pmatrix}&V_{12}&V_{13}\\V_{21}&&V_{23}\\V_{31}&V_{32}&\end{pmatrix}=i\begin{pmatrix}&Q_{12}&Q_{13}\\Q_{21}&&Q_{23}\\Q_{31}&Q_{32}&\end{pmatrix}\tag{20}$$

$$\left.\begin{aligned}A^{-1}\zeta&=-i\begin{pmatrix}Q_{11}&&\\&Q_{22}&\\&&Q_{33}\end{pmatrix},\\A&=\begin{pmatrix}\alpha_1&&\\&\alpha_2&\\&&\alpha_3\end{pmatrix}\end{aligned}\right\}\tag{21}$$

得

$$A\left[-iI\frac{\partial}{\partial x}+V\right]v=\zeta v \tag{22}$$

(22)式为 Kaup 采用的形式[5]。若 v 是(22)式的解，则 $\hat{v}=e^{-i\zeta\Delta x}v$ 也是(22)式的解，只不过用 $\hat{\alpha}_n^{-1}=\alpha_n^{-1}-\Delta$ 来代替 α_n^{-1}，故不失普遍性可约定 $\alpha_1^{-1}>\alpha_2^{-1}>\alpha_3^{-1}>0$。与 Kaup 不一样，我们按下面方式定义下半平面为解析的解 ϕ_n^j，及上半平面为解析的解 ψ_n^j，令

$$\beta_{mn}=\frac{1}{\alpha_m}-\frac{1}{\alpha_n}$$

$$\begin{aligned}
\phi_1^1 e^{-i\zeta x/\alpha_1}&=1-i\int_{-\infty}^{x}dy[V_{12}\phi_2^1 e^{-i\zeta y/\alpha_1}\\
&\quad+V_{13}\phi_3^1 e^{-i\zeta y/\alpha_1}]\\
\phi_2^1 e^{-i\zeta x/\alpha_1}&=-i\int_{-\infty}^{x}dy e^{i\zeta(x-y)\beta_{21}}\\
&\quad\cdot[V_{21}\phi_1^1 e^{-i\zeta y/\alpha_1}+V_{23}\phi_3^1 e^{-i\zeta y/\alpha_1}]\\
\phi_3^1 e^{-i\zeta x/\alpha_1}&=-i\int_{-\infty}^{x}dy\, e^{i\zeta(x-y)\beta_{31}}\\
&\quad\cdot[V_{31}\phi_1^1 e^{-i\zeta y/\alpha_1}+V_{32}\phi_2^1 e^{-i\zeta y/\alpha_1}]
\end{aligned}\tag{23}$$

$$\begin{aligned}
\psi_1^1 e^{-i\zeta x/\alpha_1}&=1+i\int_{x}^{\infty}dy[V_{12}\psi_2^1 e^{-i\zeta y/\alpha_1}\\
&\quad+V_{13}\psi_3^1 e^{-i\zeta y/\alpha_1}]\\
\psi_2^1 e^{-i\zeta x/\alpha_1}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{21}}\\
&\quad\cdot[V_{21}\psi_1^1 e^{-i\zeta y/\alpha_1}+V_{23}\psi_3^1 e^{-i\zeta y/\alpha_1}]\\
\psi_3^1 e^{-i\zeta x/\alpha_1}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{31}}\\
&\quad\cdot[V_{31}\psi_1^1 e^{-i\zeta y/\alpha_1}+V_{32}\psi_2^1 e^{-i\zeta y/\alpha_1}]
\end{aligned}\tag{24}$$

$$\begin{aligned}
\phi_1^2 e^{-i\zeta x/\alpha_2}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{12}}\\
&\quad\cdot[V_{12}\phi_2^2 e^{-i\zeta y/\alpha_2}+V_{13}\phi_3^2 e^{-i\zeta y/\alpha_2}]\\
\phi_2^2 e^{-i\zeta x/\alpha_2}&=1-i\int_{-\infty}^{x}dy[V_{21}\phi_1^2 e^{-i\zeta y/\alpha_2}\\
&\quad+V_{23}\phi_3^2 e^{-i\zeta y/\alpha_2}]\\
\phi_3^2 e^{-i\zeta x/\alpha_2}&=-i\int_{-\infty}^{x}dy\, e^{i\zeta(x-y)\beta_{32}}\\
&\quad\cdot[V_{12}\psi_2^2 e^{-i\zeta y/\alpha_2}+V_{13}\psi_3^2 e^{-i\zeta y/\alpha_2}]
\end{aligned}\tag{25}$$

$$\begin{aligned}
\psi_1^2 e^{-i\zeta x/\alpha_2}&=-i\int_{-\infty}^{x}dy\, e^{i\zeta(x-y)\beta_{12}}\\
&\quad\cdot[V_{12}\psi_2^2 e^{-i\zeta y/\alpha_2}+V_{13}\psi_3^2 e^{-i\zeta y/\alpha_2}]\\
\psi_2^2 e^{-i\zeta x/\alpha_2}&=1+i\int_{x}^{\infty}dy[V_{21}\psi_1^2 e^{-i\zeta y/\alpha_2}\\
&\quad+V_{23}\psi_3^2 e^{-i\zeta y/\alpha_2}]\\
\psi_3^2 e^{-i\zeta x/\alpha_2}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{32}}\\
&\quad\cdot[V_{31}\psi_1^2 e^{-i\zeta y/\alpha_2}+V_{32}\psi_2^2 e^{-i\zeta y/\alpha_2}]
\end{aligned}\tag{26}$$

$$\begin{aligned}
\phi_1^3 e^{-i\zeta x/\alpha_3}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{13}}\\
&\quad\cdot[V_{12}\phi_2^3 e^{-i\zeta y/\alpha_3}+V_{13}\phi_3^3 e^{-i\zeta y/\alpha_3}]\\
\phi_2^3 e^{-i\zeta x/\alpha_3}&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{23}}\\
&\quad\cdot[V_{21}\phi_1^3 e^{-i\zeta y/\alpha_3}+V_{23}\phi_3^3 e^{-i\zeta y/\alpha_3}]\\
\phi_3^3 e^{-i\zeta x/\alpha_3}&=1-i\int_{-\infty}^{x}dy[V_{31}\phi_1^3 e^{-i\zeta y/\alpha_3}\\
&\quad+V_{32}\phi_2^3 e^{-i\zeta y/\alpha_3}]
\end{aligned}\tag{27}$$

$$\begin{aligned}
\psi_1^3 e^{-i\zeta x/\alpha_3}&=-i\int_{-\infty}^{x}dy\, e^{i\zeta(x-y)\beta_{13}}\\
&\quad\cdot[V_{12}\psi_2^3 e^{-i\zeta y/\alpha_3}+V_{13}\psi_3^3 e^{-i\zeta y/\alpha_3}]\\
\psi_2^3 e^{-i\zeta x/\alpha_3}&=-i\int_{-\infty}^{x}dy\, e^{i\zeta(x-y)\beta_{23}}\\
&\quad\cdot[V_{21}\psi_1^3 e^{-i\zeta y/\alpha_3}+V_{23}\psi_3^3 e^{-i\zeta y/\alpha_3}]\\
\psi_3^3 e^{-i\zeta x/\alpha_3}&=1+i\int_{x}^{\infty}dy[V_{31}\psi_1^3 e^{-i\zeta y/\alpha_3}\\
&\quad+V_{32}\psi_2^3 e^{-i\zeta y/\alpha_3}]
\end{aligned}\tag{28}$$

方程(23)，(24)式只包含一种积分限，即 $\int_{-\infty}^{x}$ 或 $\int_{x}^{\infty}$，故其解的 Neuman 级数展开的绝对收敛性质，可按 Kaup 方法[5]证明。(25)～(28)式因包含 $\int_{-\infty}^{x}$ 与 $\int_{x}^{\infty}$ 两种积分限，其绝对收敛性证明较复杂。下面给出 (25) 式的证明，(26)～(28) 式证明仿此。为书写方便起见，我们将(25)式写成如下形状

$$\begin{aligned}
W_1(\zeta,x)&=i\int_{x}^{\infty}dy\, e^{i\zeta(x-y)\beta_{12}}\\
&\quad\cdot[V_{12}(y)W_2(\zeta,y)\\
&\quad+V_{13}(y)W_3(\zeta,y)]
\end{aligned}$$

$$W_2(\zeta, x)=1-i\int_{-\infty}^{x} dy[V_{21}(y)W_1(\zeta, y)$$

$$+V_{23}(y)W_3(\xi, y)]$$

$$W_3(\zeta, x)=-i\int_{-\infty}^{x} dy\, e^{i\zeta(x-y)\beta_{32}}$$

$$\cdot[V_{31}(y)W_1(\zeta, y)$$

$$+V_{32}(y)W_2(\zeta, y)] \tag{29}$$

又令 $\bar{\delta}_1=W_1(\zeta, x)$

$$W=\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}, \bar{\delta}=\begin{pmatrix} \bar{\delta}_1 \\ 1 \\ 0 \end{pmatrix},$$

$$\widetilde{V}=$$

$$\begin{pmatrix} 0 & 0 & 0 \\ V_{21}(y) & 0 & V_{23}(y) \\ e^{i\zeta(x-y)\beta_{32}}V_{31}(y) & e^{i\zeta(x-y)\beta_{32}}V_{32}(y) & 0 \end{pmatrix} \tag{30}$$

则(29)式可写为：

$$W(x)=\bar{\delta}+\int_{-\infty}^{x} \widetilde{V}(x, y)W(y)dy$$

$$=\Big[I+\int_{-\infty}^{x} dy_1 \widetilde{V}(x, y_1)$$

$$+\int_{-\infty}^{x} dy_1 \int_{-\infty}^{y_1} dy_2 \widetilde{V}(x, y_1)\widetilde{V}(x, y_2)$$

$$+\cdots\Big]\bar{\delta}=X\bar{\delta} \tag{31}$$

在下半平面 $\xi=s-it$, $t\geqslant 0$, 令

$$u(y)=\operatorname*{Max}_{n,m}|\widetilde{V}_{n,m}(x, y)|,$$

$$\delta^M=\mathrm{Max}|\bar{\delta}(y)|,$$

并应用等式

$$\int_{-\infty}^{x} dy_1\int_{-\infty}^{y_1} dy_2\cdots\int_{-\infty}^{y_{n-1}} dy_n u(y_1)u(y_2)\cdots u(y_n)$$

$$=\frac{1}{n!}\int_{-\infty}^{x} dy_1 \int_{-\infty}^{x} dy_2\cdots$$

$$\int_{\to\infty}^{x} dy_n u(y_1)u(y_2)\cdots u(y_n)$$

则得

$$|W(x)|\leqslant\Big(I+\overline{M}\int_{-\infty}^{x} dy_1 u(y_1)$$

$$+\frac{\overline{M}^2}{2}\int_{-\infty}^{x} dy_1 \int_{-\infty}^{a} dy_2 u(y_1)u(y_2)+\cdots\Big)\delta^M$$

$$=\Big(I+\sum_{p=1}^{\infty}\frac{1}{p!}\overline{M}^p R^p(x)\Big)\delta^M,$$

$$R(x)=\int_{-\infty}^{x} u(y)dy,$$

$$\overline{M}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\overline{M}^2=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \overline{M}^{2m+1}=\overline{M} \tag{32}$$

故当 $R(x)<\infty$ 时, $1+\sum_{p=1}^{\infty}\frac{1}{p!}\overline{M}^p R^p(x)$ 为有界, $W(x)$ 绝对收敛。将(31)代入(29)的第 1 式, 便得

$$\bar{\delta}_1=-\int_{x}^{\infty} dy\, e^{i\zeta(x-y)\beta_{12}}$$

$$\cdot[(V_{12}X_{21}+V_{13}X_{31})\bar{\delta}_1$$

$$+V_{12}X_{22}+V_{13}X_{32}] \tag{33}$$

这是一关于 $\bar{\delta}_1$ 的线性积分方程, 可用多次叠代法求解, 其收敛性证明与上同。这就证明了(29)式或(25)式的 Neuman 展开在下半平面 $\xi=s-it$ 是绝对收敛的。同样可证明(27)式的 Neuman 展开在下半平面 $\xi=s-it$ 是绝对收敛的; (26), (28)的 Neuman 展开在上半平面 $\xi=s+it$ 是绝对收敛的。故由(23), (25), (27)式定义的 $\varphi_m^j e^{-i\zeta x/\alpha_j}$ 在下半平面 $\xi=s-it$ 为解析; (24), (26), (28)式定义的 $\psi_n^j e^{-i\zeta x/\alpha_j}$ 在上半平面 $\xi=s+it$ 为解析。解(25)的步骤为先解(33)式, 后解 $W(x)$ 即(31)式。

2. 设 $g_n^j=\phi_n^j-e^{i\zeta x/\alpha_j}\delta_n^j$, 则(22)式为：

$$A\Big[-iI\frac{\partial}{\partial x}+V\Big]g^j$$

$$=\zeta g^j-AVe^{i\zeta x/\alpha_j}\delta^j$$

求 g_n^j 的 Fourier 变换

$$\hat{g}_n^j=\frac{1}{2\pi}\int_{-\infty}^{\infty} g_n^j e^{-i\zeta y/\alpha_j} d\zeta$$

为明确起见, 下面取 $j=1$

$$\begin{pmatrix} -i\alpha_1\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) & \alpha_1 V_{12} & \alpha_1 V_{13} \\ \alpha_2 V_{21} & -i\left(\alpha_2\frac{\partial}{\partial x}+\alpha_1\frac{\partial}{\partial y}\right) & \alpha_2 V_{23} \\ \alpha_3 V_{31} & \alpha_3 V_{32} & -i\left(\alpha_3\frac{\partial}{\partial x}+\alpha_1\frac{\partial}{\partial y}\right) \end{pmatrix}\begin{pmatrix} \hat{g}_1^1 \\ \hat{g}_2^1 \\ \hat{g}_2^1 \end{pmatrix}$$

$$=-AV\alpha_1\begin{pmatrix} \delta(x-y) \\ 0 \\ 0 \end{pmatrix}$$

换变数 $\xi=x+y,\ \eta=x-y,$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}$$

代入上式得

$$-i2\alpha_1\frac{\partial}{\partial \xi}\hat{g}_1^1+\alpha_1 V_{12}\hat{g}_2^1+\alpha_1 V_{13}\hat{g}_3^1=0$$

$$\alpha_2 V_{21}\hat{g}_1^1-i\Big((\alpha_1+\alpha_2)\frac{\partial}{\partial \xi}+(\alpha_2-\alpha_1)\frac{\partial}{\partial \eta}\Big)\hat{g}_2^1+\alpha_2 V_{23}\hat{g}_3^1=-\alpha_2\alpha_1 V_{21}\delta(\eta)$$

$$\alpha_3 V_{31}\hat{g}_1^1+\alpha_3 V_{32}\hat{g}_2^1-i\Big((\alpha_3+\alpha_1)\frac{\partial}{\partial \xi}+(\alpha_3-\alpha_1)\frac{\partial}{\partial \eta}\Big)\hat{g}_3^1=-\alpha_3\alpha_1 V_{31}\delta(\eta) \tag{34}$$

由 (34) 式的第 2，3 方程对 η 在 $\pm\epsilon$ 内积分并取 $\epsilon\to0^+$ 得

$$\hat{g}_2^1=\frac{i\alpha_1\alpha_2}{\alpha_1-\alpha_2}V_{21},\quad \hat{g}_3^1=\frac{i\alpha_1\alpha_3}{\alpha_1-\alpha_3}V_{31} \tag{35}$$

这里用了下面将要证明的关系 $\hat{g}_n^1=0$, 当 $\eta=x-y<0$

$$\int_{-6}^{6}\frac{\partial}{\partial \eta}\hat{g}_2^1 d\eta=\hat{g}_2^1|_{\eta=6}-\hat{g}_2^1|_{\eta=-6}=\hat{g}_2^1|_{\eta=6} \tag{36}$$

若 $\hat{G}_2^1=0$, 当 $\eta=x-y>0$, 则同样有

$$\int_{-6}^{6}\frac{\partial}{\partial \eta}\hat{G}_2^1 d\eta=\hat{G}_2^1|_{\eta=6}-\hat{G}_2^1|_{\eta=-6}=-\hat{G}_2^1|_{\eta=-6} \tag{37}$$

(37)式下面也将要用到。

将(35)式代入(34)式中的第 1 式，便得

$$2\frac{\partial}{\partial \xi}\hat{g}_1^1-\frac{\alpha_1\alpha_2}{\alpha_1-\alpha_2}V_{21}V_{12}-\frac{\alpha_1\alpha_3}{\alpha_1-\alpha_3}V_{31}V_{13}=0 \tag{38}$$

用同样方法可求得

$$\hat{g}_n^j=\frac{i\alpha_j\alpha_n}{\alpha_j-\alpha_n}V_{nj},\ j\neq n,\ \eta=\epsilon\to0^+$$

$$2\frac{\partial}{\partial \xi}\hat{g}_n^j+\sum_{j\neq n}\frac{\alpha_j V_{jn}V_{nj}\alpha_n}{\alpha_n-\alpha_j}=0 \tag{39}$$

3. 关于 g_n^j 的解析性质

按上面关于 $\hat{g}_n^j$, g_n^j 的定义及(23)～(28)式便得

$$g_n^j=\phi_n^j-\delta_n^j e^{i\zeta x/\alpha_j}$$

$$=\begin{cases} -i\int_{-\infty}^{x}dy'\,e^{i\zeta(x-y')/\alpha_n}\cdot\sum V_{nm}(y')\phi_m^j(y',\zeta) & j\leqslant n \\ i\int_{x}^{\infty}dy'\,e^{i\zeta(x-y')/\alpha_n}\cdot\sum V_{nm}(y')\phi_m^j(y',\zeta), & j>n \end{cases} \tag{40}$$

$$\hat{g}_n^j=\frac{1}{2\pi}\int_{-\infty}^{\infty}g_n^j e^{-i\zeta y/\alpha_j}d\zeta$$

$$=\begin{cases} -i\int_{-\infty}^{x}dy'\frac{1}{2\pi}\int_{-\infty}^{\infty}d\zeta\big[e^{i\zeta(x-y')\beta_{nj}}\cdot\sum V_{nm}\phi_m^j e^{-i\zeta y'/\alpha_j}\big]e^{i\zeta(x-y)/\alpha_j} & j\leqslant n \\ i\int_{x}^{\infty}dy'\frac{1}{2\pi}\int_{-\infty}^{\infty}d\zeta\big[e^{i\zeta(x-y')\beta_{nj}}\cdot\sum V_{nm}\phi_m^j e^{-i\zeta y'/\alpha_j}\big]e^{i\zeta(x-y)/\alpha_j} & j>n \end{cases} \tag{41}$$

已经证明 $\sum V_{nm}(y')\phi_m^j(y',\zeta)e^{-i\zeta y'/\alpha_j}$ 在 ζ 的下半平面为解析的。则在半圆 R 上，当 $n\neq j$ 时，$e^{i\zeta(x-y)\beta_{nj}}=e^{is(x-y)\beta_{nj}}\times e^{t(x-y')\beta_{nj}}$，按(41)，$(x-y')\beta_{nj}<0$，故积分表式中方括号在半圆 R 上随同因子 $e^{t(x-y')\beta_{nj}}$ 一致地趋于零。当 $n=j$，按即将证明的(56)式，

$$\lim_{\zeta\to\infty}\phi_m^j e^{-i\zeta y'/\alpha_j}=\delta_m^j=0。$$

故当 R 选得足够大时，总可以使方括号中函数的绝对值$<\epsilon$，于是按 Jordan 予理[9]

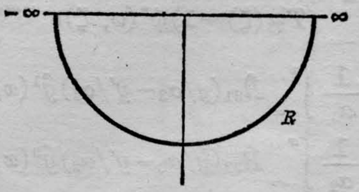

图 1

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\right|\leqslant\left|\frac{\epsilon}{2\pi}\int_{R}e^{i\zeta(x-y)/\alpha_j}d\zeta\right|$$

$$<\left|\frac{\epsilon}{2\pi}\int_{0}^{\pi}e^{'(x-y)/\alpha_j}d\zeta\right|$$

设 $\frac{y-x}{\alpha_j}>0$，则上式为

$$\frac{\epsilon}{\pi}\int_{0}^{\pi/2}\rho e^{-(y-x)\rho\sin\theta/\alpha_j}d\theta\leqslant\frac{\epsilon}{2\left(\frac{y-x}{\alpha_j}\right)} \tag{42}$$

故当 $\frac{y-x}{\alpha_j}>0$ 时，

$$\hat{g}_n^j(x, y)=0 \tag{43}$$

$\hat{g}_n^j$ 求得后，便可由(39)式求得 V_{nj}。

完全同样，定义

$$G_n^j=\psi_n^j-\delta_n^j e^{i\zeta x/\alpha_j} \tag{44}$$

$$\hat{G}_n^j=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_n^j e^{-i\zeta y/\alpha_j}d\zeta \tag{45}$$

由于 $\sum V_{n,m}(y')\psi_m^j(y', \zeta)e^{-\zeta y'/\alpha_j}$ 在上半平面 $(\zeta=s+it, t\geqslant0)$ 为解析。完全同样可证明

当 $\frac{y-x}{\alpha_j}<0$，

$$\hat{G}_n^j(x, y)=0 \tag{46}$$

4．积分方程

按照(23)～(28)，$\phi^j(j=1, 2, 3)$ 是一组独立解。而 ψ^j 可用 ϕ^j 表示为

$$\psi^j(x, \zeta)=\sum a_{jk}\phi^k(x, \zeta) \tag{47}$$

取 $j=1$ 为例，便得

$$\frac{1}{a_{11}}\psi^1-\frac{a_{12}}{a_{11}}\phi^2-\frac{a_{13}\phi^3}{a_{11}}=\phi^1 \tag{48}$$

令

$$\left.\begin{aligned}T_{11}=1/a_{11},\ R_{12}=-a_{12}/a_{11},\\ R_{13}=-a_{13}/a_{11}\end{aligned}\right\} \tag{49}$$

便有

$$T_{11}\psi^1+R_{12}\phi^2+R_{13}\phi^3=\phi^1 \tag{50}$$

当 ζ→∞ 时，可证明 $\lim\limits_{\zeta\to\infty}T_{11}(\zeta)=1$。实际上由(50)式得

$$\left.\begin{aligned}T_{11}=\frac{W(\phi^1, \phi^2, \phi^3)}{W(\psi^1, \phi^2, \phi^3)},\\ W(\phi^1, \phi^2, \phi^3)=\det\begin{vmatrix}\phi_1^1 & \phi_1^2 & \phi_1^3\\ \phi_2^1 & \phi_2^2 & \phi_2^3\\ \phi_3^1 & \phi_3^2 & \phi_3^3\end{vmatrix}\end{aligned}\right\} \tag{51}$$

$$\frac{\partial}{\partial x}W(\phi^1, \phi^2, \phi^3)=i\zeta(1/\alpha_1+1/\alpha_2+1/\alpha_3)W(\phi^1, \phi^2, \phi^3) \tag{52}$$

由(23)，(25)，(27)式得

$$\lim_{x\to-\infty}\phi_n^j e^{-i\zeta x/\alpha_j}=\delta_n^j \quad 当\ j\leqslant n \tag{53}$$

由(52)，(53)式得

$$W(\phi^1, \phi^2, \phi^3)=e^{i\zeta x(1/\alpha_1+1/\alpha_2+1/\alpha_3)} \tag{54}$$

代入(51)式得

$$T_{11}(\zeta)=\frac{1}{W(\psi^1 e^{-i\zeta x/\alpha_1}, \phi^2 e^{-i\zeta x/\alpha_2}, \phi^3 e^{-i\zeta x/\alpha_3})} \tag{55}$$

按(23)～(28)式及 Rieman 予备定理[10]得

$$\lim_{\zeta\to\infty}\psi_n^j e^{-i\zeta x/\alpha_j}=\lim_{\zeta\to\infty}\phi_n^j e^{-i\zeta x/\alpha_j}=\delta_n^j \tag{56}$$

由(55)，(56)式得

$$\lim_{\zeta\to\infty}T_{11}(\zeta)=1 \tag{57}$$

积分方程是在(50)式基础上建立的。

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(T_{11}(\zeta)-1)\psi^1(x, \zeta)e^{-i\zeta y/\alpha_1}d\zeta$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{12}(\zeta)\left(\phi^2(x,\zeta)-\begin{pmatrix}0\\1\\0\end{pmatrix}e^{i\zeta x/\alpha_2}\right)$$

$$\times e^{-i\zeta y/\alpha_1}d\zeta+\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{13}(\zeta)$$

$$\cdot\left(\phi^3(x, \zeta)-\begin{pmatrix}0\\0\\1\end{pmatrix}e^{i\zeta x/\alpha_3}\right)e^{-i\zeta y/\alpha_1}d\zeta$$

$$+\frac{1}{2\pi}\begin{pmatrix}0\\1\\0\end{pmatrix}\int_{-\infty}^{\infty}R_{12}(\zeta)e^{i\zeta(x/\alpha_2-y/\alpha_1)}d\zeta$$

$$+\frac{1}{2\pi}\begin{pmatrix}0\\0\\1\end{pmatrix}\int_{-\infty}^{\infty}R_{13}(\zeta)e^{i\zeta(x/\alpha_3-y/\alpha_1)}d\zeta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\psi^1(x, \zeta)+\begin{pmatrix}1\\0\\0\end{pmatrix}e^{i\zeta x/\alpha_1}\right.$$

$$-\begin{pmatrix}1\\0\\0\end{pmatrix}e^{i\zeta x/\alpha_1}+\phi^1(x,\ \zeta)\Bigg]e^{-i\zeta y/\alpha_1}d\zeta \quad (58)$$

令
$$\hat{R}_{12}(y/\alpha_1-y'/\alpha_2)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{12}(\zeta)e^{-i\zeta(y/\alpha_1-y'/\alpha_2)}d\zeta$$
$$\hat{R}_{13}(y/\alpha_1-y'/\alpha_3)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{13}(\zeta)e^{-i\zeta(y/\alpha_1-y'/\alpha_3)}d\zeta \quad (59)$$

参照(43)，(45)，(46)便可将(58)式写为

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(T_{11}(\zeta)-1)\psi^1(x,\ \zeta)e^{-i\zeta y/\alpha_1}d\zeta$$
$$+\frac{1}{\alpha_2}\int_{-\infty}^{x}\hat{R}_{12}(y/\alpha_1-y'/\alpha_2)\hat{g}^2(x,\ y')dy'$$
$$+\frac{1}{\alpha_3}\int_{-\infty}^{x}\hat{R}_{13}(y/\alpha_1-y'/\alpha_3)\hat{g}^3(x,\ y')dy'$$
$$+\begin{pmatrix}0\\1\\0\end{pmatrix}\hat{R}_{12}(y/\alpha_1-x/\alpha_2)$$
$$+\begin{pmatrix}0\\0\\1\end{pmatrix}\hat{R}_{13}(y/\alpha_1-x/\alpha_3)$$
$$=\hat{g}^1(x,\ y)-\hat{G}^1(x,\ y)$$
$$=\hat{g}^1(x,\ y)\quad x>y \quad (60)$$

同样可证

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(T_{22}(\zeta)-1)\psi^2(x,\ \zeta)e^{-i\zeta y/\alpha_2}d\zeta$$
$$+\frac{1}{\alpha_1}\int_{-\infty}^{x}\hat{R}_{21}(y/\alpha_2-y'/\alpha_1)\hat{g}^1(x,\ y')dy'$$
$$+\frac{1}{\alpha_3}\int_{-\infty}^{x}\hat{R}_{23}(y/\alpha_2-y'/\alpha_3)\hat{g}^3(x,\ y')dy'$$
$$+\begin{pmatrix}1\\0\\0\end{pmatrix}\hat{R}_{21}(y/\alpha_2-x/\alpha_1)$$
$$+\begin{pmatrix}0\\0\\1\end{pmatrix}\hat{R}_{23}(y/\alpha_2-x/\alpha_3)$$
$$=\hat{g}^2(x,\ y)-\hat{G}^2(x,\ y)$$
$$=\hat{g}^2(x,\ y)\quad x>y \quad (61)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(T_{33}(\zeta)-1)\psi^3(x,\ \zeta)e^{-i\zeta y/\alpha_3}d\zeta$$
$$+\frac{1}{\alpha_1}\int_{-\infty}^{x}\hat{R}_{31}(y/\alpha_3-y'/\alpha_1)\hat{g}^1(x,\ y')dy'$$
$$+\frac{1}{\alpha_2}\int_{-\infty}^{x}\hat{R}_{32}(y/\alpha_3-y'/\alpha_2)\hat{g}^2(x,\ y')dy'$$
$$+\begin{pmatrix}1\\0\\0\end{pmatrix}\hat{R}_{31}(y/\alpha_3-x/\alpha_1)$$
$$+\begin{pmatrix}0\\1\\0\end{pmatrix}\hat{R}_{32}(y/\alpha_3-x/\alpha_2)$$
$$=\hat{g}^3(x,\ y)-\hat{G}^3(x,\ y)$$
$$=\hat{g}^3(x,\ y)\quad x>y \quad (62)$$

(60)～(62)是关于$\hat{g}^1(x,\ y)$，$\hat{g}^2(x,y)$，$\hat{g}^3(x,\ y)$的联立方程组。这里利用了$1/\alpha_1>1/\alpha_2>1/\alpha_3>0$及(46)式。当$\hat{g}^j(x,\ y)$求得后，就可用(39)式求$V_{nj}$，上面三个积分方程是就$x>y$解$\hat{g}^j(x,\ y)$，也可就$y>x$情形解$G^j(x,\ y)$，积分方程组的形状与上同。

5．积分方程的进一步简化

已证$\lim\limits_{\zeta\to\infty}T_{11}(\zeta)=1$，故将$T_{11}(\zeta)-1$展开成$\zeta+iK_n$的级数时不包括常数项。现研究$T_{11}(\zeta)-1$的极点。由(55)式得出$T_{11}(\zeta)$的如下表示式

$$T_{11}(\zeta)=\frac{1}{W(\psi^1e^{-i\zeta x/\alpha_1},\phi^2e^{-i\zeta x/\alpha_2},\phi^3e^{-i\zeta x/\alpha_3})} \quad (63)$$

参照(51)，(52)易证

$$\frac{\partial}{\partial x}W(\psi^1e^{-i\zeta x/\alpha_1},\ \phi^2e^{-i\zeta x/\alpha_2},\ \phi^3e^{-i\zeta x/\alpha_3})=0 \quad (64)$$

故$W(\psi^1e^{-i\zeta x/\alpha_1},\phi^2e^{-i\zeta x/\alpha_2},\ \phi^3e^{-i\zeta x/\alpha_3})$的值与x无关。零点$\zeta=-iK_n$也与$x$无关。$T_{11}(\zeta)-1$留数计算见附录2．(60)式第1项可写成如下形状

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(T_{11}(\zeta)-1)\psi^1(x,\ \zeta)e^{-i\zeta y/\alpha_1}d\zeta$$
$$=\sum_n m_{2n}e^{-K_{2n}y/\alpha_1}\phi^2(x_1-iK_{2n})$$
$$+\sum_n m_{3n}e^{-K_{3n}y/\alpha_1}\phi^3(x,\ -iK_{3n}) \quad (65)$$

应用(40)～(43)式并设

$$\hat{F}_{12}(x,\ y)=\hat{R}_{12}(y/\alpha_1-x/\alpha_2)+\sum m_{2n}e^{-K_{2n}(y/\alpha_1-x/\alpha_2)}$$

$$\hat{F}_{13}(x,\ y)=\hat{R}_{13}(y/\alpha_1-x/\alpha_3)+\sum m_{3n}e^{-K_{3n}(y/\alpha_1--x/\alpha_3)} \qquad (66)$$

则有

$$\frac{1}{\alpha_2}\int_{-\infty}^{x}\hat{F}_{12}(y',\ y)\hat{g}^2(x,\ y')dy'+\frac{1}{\alpha_3}\int_{-\infty}^{x}\hat{F}_{13}(y',\ y)\hat{g}^3(x,\ y')dy'+\begin{pmatrix}0\\1\\0\end{pmatrix}\hat{F}_{12}(x,\ y)+\begin{pmatrix}0\\0\\1\end{pmatrix}\hat{F}_{13}(x,\ y)=\hat{g}^1(x,\ y),\ \ x>y \qquad (67)$$

同样

$$\hat{F}_{21}(x,\ y)=\hat{R}_{21}(y/\alpha_2-x/\alpha_1)+\sum m_{1n}e^{-K_{1n}(y/\alpha_2-x/\alpha_1)}$$

$$\hat{F}_{23}(x,\ y)=\hat{R}_{23}(y/\alpha_2-x/\alpha_3)+\sum m_{3n}e^{-K_{3n}(y/\alpha_2-x/\alpha_3)} \qquad (68)$$

$$\frac{1}{\alpha_1}\int_{-\infty}^{x}\hat{F}_{21}(y',\ y)\hat{g}^1(x,\ y')dy'+\frac{1}{\alpha_3}\int_{-\infty}^{x}\hat{F}_{23}(y',\ y)\hat{g}^3(x,\ y')dy'+\begin{pmatrix}1\\0\\0\end{pmatrix}\hat{F}_{21}(x,\ y)+\begin{pmatrix}0\\0\\1\end{pmatrix}\hat{F}_{23}(x,\ y)=\hat{g}^2(x,\ y),\ \ x>y \qquad (69)$$

$$\hat{F}_{31}(x,\ y)=\hat{R}_{31}(y/\alpha_3-x/\alpha_1)+\sum m_{1n}e^{-K_{1n}(y/\alpha_3-x/\alpha_1)}$$

$$\hat{F}_{32}(x,\ y)=\hat{R}_{32}(y/\alpha_3-x/\alpha_2)+\sum m_{2n}e^{-K_{2n}(y/\alpha_3-x/\alpha_1)} \qquad (70)$$

$$\frac{1}{\alpha_1}\int_{-\infty}^{x}\hat{F}_{31}(y',\ y)\hat{g}^1(x,\ y')dy'+\frac{1}{\alpha_2}\int_{-\infty}^{x}\hat{F}_{32}(y',\ y)\hat{g}^2(x,\ y')dy'+\begin{pmatrix}1\\0\\0\end{pmatrix}\hat{F}_{31}(x,\ y)+\begin{pmatrix}0\\1\\0\end{pmatrix}\hat{F}_{32}(x,\ y)=\hat{g}^3(x,\ y)\quad x>y \qquad (71)$$

对于 $y>x$ 有相应的关于 $\hat{G}^1$, $\hat{G}^2$, $\hat{G}^3$ 的方程组，形状与(66)～(71)式相同。

6．n 阶波相互作用的逆散射问题

上面虽是就三阶波相互作用进行讨论的，但推广到 n 阶波是没有困难的。现逐一看各式的推广。

(a) (23)～(28)式的推广

$$\varphi_n e^{-i\zeta x/\alpha_j}=\delta_n^j\begin{cases}-i\int_{-\infty}^{x} & j\leqslant n\\ +i\int_{x}^{\infty} & j>n\end{cases}dy\,e^{i\zeta(x-y)\beta_{nj}}\sum V_{nm}\phi_m^j e^{-i\zeta x/\alpha_j}$$

$$\psi_n^j e^{-i\zeta x/\alpha_j}=\delta_n^j\begin{cases}+i\int_{x}^{\infty} & j\leqslant n\\ -i\int_{-\infty}^{x} & j>n\end{cases}dy\,e^{i\zeta(x-y)\beta_{nj}}\sum V_{nm}\psi_m^j e^{-i\zeta x/\alpha_j} \qquad (72)$$

(72)式的 Neuman 展开的收敛性证明在附录1中给出。

(b) (39)、(43)、(46)式本身已写成推广了的形式。从(72)出发就能得出这些推广了的结果。

(c) (54)、(56)式可推广为

$$W(\phi^1,\ \phi^2,\ \cdots\phi^n)=e^{i\zeta x\sum_1^n\alpha_m^{-1}}$$

$$\lim_{\zeta\to\infty}\psi_n^j e^{-i\zeta x/\alpha_j}=\lim_{\zeta\to\infty}\phi_n^j e^{-i\zeta x/\alpha_j}=\delta_n^j \qquad (73)$$

(d) (66)～(71)式的推广形式为

$$\hat{F}_{mn}(x,\ y)=\hat{R}_{mn}(y/\alpha_m-x/\alpha_n)+\sum m_{nn'}e^{-K_{nn'}(y/\alpha_m-x/\alpha_n)}$$

$$\sum_{m\neq j}\frac{1}{\alpha_m}\int_{-\infty}^{x}\hat{F}_{jm}(y',\ x)\hat{g}^m(x,\ y')dy'+\sum_{j\neq j}\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\hat{F}_{jm}(x,\ y)=\hat{g}^j(x,\ y) \qquad (74)$$

附　录　1

由(72)式定义的 φ^j、ψ^j 收敛性证明

参照前面对(23)～(28)式收敛性证明，我们将证明分为两部分。第一部分是只包含一种积分限 $\int_{-\infty}^{x}$

或 $\int_x^\infty$ 的积分方程组，这证明可参照 Kaup[5] 方法求得

$$|W(x)| \leqslant \left[I+\sum_{p=1}^{\infty}\frac{1}{p!}M^pR^p(x)\right]\delta \quad A(1.1)$$

式中 $R(x)$ 的定义同(32)式，δ 为 n 行的列矩阵。M 为

$$M=\begin{pmatrix} 0 & 1 & 1\cdots1 \\ 1 & 0 & 1\cdots1 \\ 1 & 1 & 0\cdots1 \\ \vdots & \vdots & \vdots\quad\vdots \\ 1 & 1 & 1\quad 0 \end{pmatrix} \quad A(1.2)$$

由 A(1.2)易证

$$M^2=(n-1)I+(n-2)M \quad A(1.3)$$

设

$$M^p=a_pI+b_pM \quad A(1.4)$$

则由 A(1.3)，A(1.4)及 $M^{p+1}=M^p\cdot M$ 得

$$a_{p+1}=b_p(n-1),\ b_{p+1}=a_p+b_p(n-2) \quad A(1.5)$$

由 A(1.5)的两式相减，并考虑到 $a_1=0,\ b_1=1$,

$$a_{p+1}-b_{p+1}=b_p-a_p=(-1)^{p+1} \quad A(1.6)$$

代入 A(1.5)得

$$b_{p+1}=(-1)^p+b_p(n-1) \quad A(1.7)$$

由此得

$$b_{p+1}=(-1)^p\frac{1-(1-n)^{p+1}}{n} \quad A(1.8)$$

$$a_{p+1}=(-1)^p\frac{1-n-(1-n)^{p+1}}{n} \quad A(1.9)$$

当 $n=3$，$b_{p+1}=(-1)^p\dfrac{1-(-2)^{p+1}}{3}$

$$a_{p+1}=(-1)^p\frac{-2-(-2)^{p+1}}{3} \quad A(1.10)$$

A(1.10)式与 Kuap[5] 给出的 $M^p=2I+(2p-3)M$ 不符，易证 Kaup 的结果是错的。

$$\begin{aligned} M^{p+1}&=(2I+(2p-3)M)M \\ &=2M+2(p-3)(2I+M) \\ &=2(2p-3)I+(2(p+1)-3)M \\ &\neq 2I+(2(p+1)-3)M \end{aligned}$$

不能自恰。

由 A(1.4)，A(1.8)，A(1.9)式，故当 $R(\infty)<\infty$ 时，A(1.1)为有界的，收敛性得证。

现讨论证明的第二部分即积分方程中包括 $\int_{-\infty}^x$ 与 $\int_x^\infty$ 两种积分限。这时可参照(29)～(32)式取定

$$\delta_i=W_i(\zeta,\ x) \quad 当\ i<j$$

$$\delta_i=1,\ \delta_s=0, \quad 当\ i>j$$

$\tilde{V}$ 也可类似地予以推广，并得出相应于(31)，(32)的表式，不同的是(32)式中的 $\overline{M}$ 由下式定义。

$$\overline{M}=\left(\begin{array}{cccc} 0\cdots0 & 0\cdots & & 0 \\ \vdots\quad\vdots & \vdots & & \vdots \\ 0\cdots0 & 0 & & 0 \\ 1\cdots1 & 0 & 1\cdots1 & \\ \vdots\quad\vdots & 1 & 0\cdots1 & \\ & \vdots & & \\ \underbrace{1\cdots1}_{m} & \multicolumn{3}{c}{\underbrace{1\quad 1\cdots0}_{n-m}} \end{array}\right)\begin{array}{l}\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}m \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ \vdots\\ 1\end{matrix}}\right\}n-m\end{array} =N+M,$$

$$M=\left(\begin{array}{cccc} 0\cdots0\cdots & & & 0 \\ \vdots\quad\vdots & & & \vdots \\ 0\cdots0 & 0 & & \\ 0\cdots0 & 0 & 1\cdots1 & \\ \vdots\quad\vdots & 1 & 0\cdots1 & \\ & \vdots & \vdots & \\ \underbrace{0\cdots0}_{m} & \multicolumn{3}{c}{\underbrace{1\quad 1\quad 0}_{n-m}} \end{array}\right)\begin{array}{l}\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}m \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ \vdots\\ 1\end{matrix}}\right\}n-m\end{array}$$

$$N=\left(\begin{array}{cc} 0\cdots0 & 0\quad 0 \\ \vdots\quad\vdots & \vdots\quad\vdots \\ 0\cdots0 & 0\cdots0 \\ 1\cdots1 & 0\cdots0 \\ \vdots\quad\vdots & \vdots\quad\vdots \\ \underbrace{1\cdots1}_{m} & \underbrace{0\cdots0}_{n-m} \end{array}\right)\begin{array}{l}\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}m \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\}n-m\end{array}$$

$$I=\left(\begin{array}{cccc} 0\cdots0 & 0\cdots & & 0 \\ \vdots\quad\vdots & \vdots & & \vdots \\ 0\cdots0 & 0\cdots & & 0 \\ 0\cdots0 & 1 & 0\cdots & 0 \\ \vdots\quad\vdots & 0 & 1 & \vdots \\ & \vdots & \vdots & \\ \underbrace{0\cdots0}_{m} & \multicolumn{3}{c}{\underbrace{0\quad 0\quad 1}_{n-m}} \end{array}\right)\begin{array}{l}\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\}m \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ \vdots\\ 1\end{matrix}}\right\}n-m\end{array} \quad A(1.11)$$

由 A(1.11)，便能计算出

$$N\overline{M}=0,\ \overline{M}^2=M\overline{M}$$

$$\overline{M}^{p+1}=M^p\overline{M}=M^pN+M^{p+1} \quad A(1.12)$$

$$M^p=a_pI+b_pM$$

$$\left.\begin{aligned} b_{p+1}&=(-1)^p\frac{1-(1-n+m)^{p+1}}{n-m}, \\ a_{p+1}&=(-1)^p\frac{1-n+m-(1-n+m)^{p+1}}{n-m} \end{aligned}\right\} \quad A(1.13)$$

应用 A(1.13)，A(1.12)式，将 $\overline{M}^{p+1}$ 代入相应于(32)的表式中，当 $R(\infty)<\infty$，$|X|$ 绝对收敛。仿照(33)式，将 $W=X\delta$ 代入 $\delta_i=W_i(\zeta,\ x)\ (i<j)$ 中，便得出 $\delta_i(i<j)$ 的积分方程组。这方程组仅包含 $\int_x^\infty$ 一种积分限，故可按在上面推广了的 Kaup 方法 A(1.1)～A(1.10)证明其收敛性。求解步骤也是先解 δ_i，后解 W。

附 录 2

$T_{11}(\zeta)-1$ 的留数计算

设 u、v、w 分别代表 ψ^1、ϕ^2、ϕ^3、并为下面方程的解

$$-i\frac{\partial}{\partial x}u_1+V_{12}u_2+V_{13}u_3=\frac{\zeta}{\alpha_1}u_1$$

$$V_{21}u_1-i\frac{\partial}{\partial x}u_2+V_{23}u_3=\frac{\zeta}{\alpha_2}u_2 \qquad \mathrm{A}(2.1)$$

$$V_{31}u_1+V_{32}u_2-i\frac{\partial}{\partial x}u_3=\frac{\zeta}{\alpha_3}u_3$$

$$W(u(x,\ \zeta),\ v(x,\ \zeta),\ w(x,\ -iK_n))$$

$$=\det\begin{vmatrix} u_1(x,\ \zeta) & v_1(x,\ \zeta) & w_1(x,\ -iK_n) \\ u_2(x,\ \zeta) & v_2(x,\ \zeta) & w_2(x,\ -iK_n) \\ u_3(x,\ \zeta) & v_3(x,\ \zeta) & w_3(x,\ -iK_n) \end{vmatrix}$$

则

$$-i\frac{\partial W}{\partial x}$$

$$=\det\begin{vmatrix} (\zeta/\alpha_1)u_1-V_{12}u_2-V_{13}u_3 & v_1 & w_1 \\ (\zeta/\alpha_2)u_2-V_{23}u_3-V_{21}u_1 & v_2 & w_1 \\ (\zeta/\alpha_3)u_3-V_{31}u_1-V_{32}u_2 & v_3 & w_3 \end{vmatrix}$$

$$+\det\begin{vmatrix} u_1 & (\zeta/\alpha_1)v_1-V_{12}v_2-V_{13}v_3 & w_1 \\ u_2 & (\zeta/\alpha_2)v_2-V_{23}v_3-V_{21}v_1 & w_2 \\ u_3 & (\zeta/\alpha_3)v_3-V_{31}v_1-V_{32}v_2 & w_3 \end{vmatrix}$$

$$+\det\begin{vmatrix} u_1 & v_1 & \frac{-iK_n}{\alpha_1}w_1-V_{12}w_2-V_{13}w_3 \\ u_2 & v_2 & \frac{-iK_n}{\alpha_2}u_2-V_{23}w_3-V_{21}w_1 \\ u_3 & v_3 & \frac{-iK_n}{\alpha_3}w_3-V_{31}w_1-V_{32}w_2 \end{vmatrix}$$

$$=\zeta\frac{W}{\alpha}+\det\begin{vmatrix} u_1 & v_1 & \frac{-iK_n-\zeta}{\alpha_1}w_1 \\ u_2 & v_2 & \frac{-iK_n-\zeta}{\alpha_2}w_2 \\ u_3 & v_3 & \frac{-iK_n-\zeta}{\alpha_3}w_3 \end{vmatrix},$$

$$1/\alpha=1/\alpha_1+1/\alpha_2+1/\alpha_3 \qquad \mathrm{A}(2.3)$$

由此得

$$\frac{\partial}{\partial x}(We^{-i\zeta x/\alpha})=-i\det\begin{vmatrix} u_1 & v_1 & w_1/\alpha_1 \\ u_2 & v_2 & w_2/\alpha_2 \\ u_3 & v_3 & w_3/\alpha_3 \end{vmatrix}$$

$$\cdot(\zeta+iK_n)e^{-i\zeta x/\alpha} \qquad \mathrm{A}(2.4)$$

$$\frac{\partial}{\partial\zeta}(We^{-i\zeta\alpha/\alpha})\Big|_{x_0}^{x}\Big|_{\zeta=-iK_n}$$

$$=-i\int_{x_0}^{x}\det\begin{vmatrix} u_1 & v_1 & w_1/\alpha_1 \\ u_2 & v_2 & w_2/\alpha_2 \\ u_3 & v_3 & w_3/\alpha_3 \end{vmatrix}e^{-i\zeta x'/\alpha}dx'\Big|_{\zeta=-iK_n}$$

$$\mathrm{A}(2.5)$$

$$\lim_{\zeta\to-iK_n}\frac{\left\{\begin{matrix} W(u(x_2,\ \zeta),\ v(x_2,\ \zeta),\ w(x_2,\ \zeta))e^{-i\zeta x_2/\alpha} \\ -W(u(x_1,\ -iK_n),\ v(x_1,\ -iK_n), \\ w(x_1,\ -iK_n))e^{-i\zeta x_1/\alpha'''} \end{matrix}\right\}}{\zeta+iK_n}$$

$$=\lim_{\zeta\to-iK_n}\frac{A_1+A_2+A_3+A_4}{\zeta+iK_n} \qquad \mathrm{A}(2.6)$$

$$A_1=W(u(x_2,\ \zeta),\ v(x_2,\ \zeta),\ w(x_2,\ \zeta))e^{-i\zeta x_2/\alpha}$$
$$-W(u(x_2,\ -iK_n),$$
$$v(x_2,\ \zeta),\ w(x_2,\ \zeta))e^{-i\zeta x_2/\alpha'}$$

$$A_2=W(u(x_2,\ -iK_n),v(x_2,\ \zeta),w(x_2,\ \zeta))e^{-i\zeta\frac{x}{\alpha}}$$
$$-W(u(x_1^0,\ -iK_n),$$
$$v(x_1,\ \zeta),\ w(x_1,\ \zeta))e^{-i\zeta x_1/\alpha'}$$

$$A_3=W(u(x_1,\ -iK_n),\ v(x_1,\ \zeta),\ w(x,\ \zeta))e^{-i\zeta x_1/\alpha}$$
$$-W(u(x_1,\ -iK_n),\ v(x_1,\ -iK_n),$$
$$w(x_1,\ \zeta))e^{-i\zeta x_1/\alpha''}$$

$$A_4=W(u(x_1,\ -iK_n),\ v(x_1,$$
$$-iK_n),w(x_1,\zeta))e^{-i\zeta x_1/\alpha''}-W(u(x_1,-iK_n),$$
$$v(x_1,\ -iK_n),\ w(x_1,\ -iK_n))e^{-i\zeta x/\alpha'''}$$

$$\zeta/\alpha'=\frac{-iK_n}{\alpha_1}+\frac{\zeta}{\alpha_2}+\frac{\zeta}{\alpha_3},$$

$$\zeta/\alpha''=-iK_n/\alpha_1-iK_n/\alpha_2+\zeta/\alpha_3$$

$$\zeta/\alpha'''=-iK_n/\alpha_1-i\frac{K_n}{\alpha_2}-i\frac{K_n}{\alpha_3}。$$

$$\mathrm{A}(2.7)$$

由 A(2.5)式得

$$\lim_{\zeta\to-iK_n}\frac{A_2}{\zeta+iK_n}$$

$$=-i\int_{x_1}^{x_2}\det\begin{vmatrix} u_1/\alpha_1 & v_1 & w_1 \\ u_2/\alpha_2 & v_2 & w_2 \\ u_3/\alpha_3 & v_3 & w_3 \end{vmatrix}e^{-i\zeta x'/\alpha'}dx'\Big|_{\zeta=-iK_n}$$

由 $W(u(x,\ -iK_n),v(x,\ -iK_n),w(x,\ -iK_n))=0$

得出 $ue^{-K_nx/\alpha_1}=c_1ve^{-K_nx/\alpha_2}+c_2we^{-K_nx/\alpha_3}$

$$\lim_{\zeta\to-iK_n}\frac{A_2}{\zeta+iK_n}$$

$$=-i\left(c_1\int_{x_1}^{x_2}\det\begin{vmatrix} v_1/\alpha_1 & v_1 & w_1 \\ v_2/\alpha_2 & v_2 & w_2 \\ v_3/\alpha_3 & v_3 & w_3 \end{vmatrix}e^{-i\zeta x'/\alpha_2}dx'\right.$$

$$\left.+c_2\int_{x_1}^{x_2}\det\begin{vmatrix} w_1/\alpha_1 & v_1 & w_1 \\ w_2/\alpha_2 & v_2 & w_2 \\ w_3/\alpha_3 & v_3 & w_3 \end{vmatrix}e^{-i\zeta x'/\alpha_3}dx'\right)_{\zeta=-iK_n}$$

$$\mathrm{A}(2.8)$$

注意到 u、v、w 分别代表 ψ^1、ϕ^2、ϕ^3，而 $\phi_n^j e^{-i\zeta x/\alpha_j}$ 在 ζ 的下半平面为解析的。

$$\phi_1^3e^{-i\zeta x/\alpha_3}=i\int_x^\infty dy\, e^{i\zeta(x-y)\beta_{13}}[V_{12}\phi_2^3e^{-i\zeta y/\alpha_3}$$

$$+V_{13}\phi_3^3e^{-i\zeta y/\alpha_3}] \qquad \mathrm{A}(2.9)$$

当 $x\to-\infty$ 时，可将上面积分写为

$$\int_{x}^{\infty}=\int_{x}^{-N}+\int_{-N}^{\infty}$$

按 Neuman 展开绝对收敛，上面积分存在的必要条件为对任意给定的 ϵ，均可求得 N，使得 $\int_{x}^{-N}<\epsilon$，另一方面，按 A(2.9)式当 $\zeta\to-iK_n$ 时

$$\int_{-N}^{\infty} e^{i\zeta\beta_{13}(x-y)}[\ ]dy$$

$$=e^{K_n\beta_{13}(x+N)}\int_{-N}^{\infty}e^{-K_n\beta_{13}(N+y)}[\ ]dy \qquad \text{A}(2.10)$$

当 $x+N$ 足够负时，上面积分 A(2.10)可任意地小，这就证明了

$$\lim_{x\to-\infty}\phi_1^3e^{-i\zeta x/\alpha_3}=0 \qquad \text{A}(2.11)$$

同样可证

$$\lim_{x\to-\infty}\phi_2^3e^{-i\zeta x/\alpha_3}=0 \qquad \text{A}(2.12)$$

按(29)式

$$\lim_{x\to-\infty}\phi_3^3e^{-i\zeta x/\alpha_3}=1 \qquad \text{A}(2.13)$$

由 A(2.11)～A(2.13) 得

$$\lim_{\zeta\to-iK_n}\lim_{x\to-\infty}\frac{\phi_n^3(x,\ \zeta)e^{-i\zeta x/\alpha_3}-\phi_n^3(x,\ -iK_n)e^{-K_nx/\alpha_3}}{\zeta+iK_n}$$

$$=\frac{d}{d\zeta}\Big[\lim_{x\to-\infty}\phi_n^3(x,\ \zeta)e^{-i\zeta x/\alpha_3}\Big]_{\zeta=-iK_n}=0 \qquad \text{A}(2.14)$$

由 A(2.7)、A(2.14)得

$$\lim_{\zeta\to-iK_n}\lim_{x\to-\infty}\frac{A_4}{\zeta+iK_n}=0 \qquad \text{A}(2.15)$$

同理可证

$$\frac{d}{d\zeta}\Big[\lim_{x\to-\infty}\phi_n^2(x,\ \zeta)e^{-i\zeta x/\alpha_2}\Big]_{\zeta=-iK_n}=0 \qquad \text{A}(2.16)$$

$$\lim_{\zeta\to-iK_n}\lim_{x\to-\infty}\frac{A_3}{\zeta+iK_n}=0 \qquad \text{A}(2.17)$$

由 A_1 的定义及 A(2.11)～A(2.13)，A(2.16)式得

$$\lim_{x_2\to\infty}\lim_{\zeta\to-iK_n}\frac{A_1}{\zeta+iK_n}$$

$$=\det\left|\begin{matrix}\frac{\partial}{\partial\zeta}(\psi_1^1e^{-i\zeta x/\alpha_1}) & 0 & 0\\ \frac{\partial}{\partial\zeta}(\psi_2^1e^{-i\zeta x/\alpha_1}) & 1 & 0\\ \frac{\partial}{\partial\zeta}(\psi_3^1e^{-i\zeta x/\alpha_1}) & 0 & 1\end{matrix}\right|_{\substack{\zeta=-iK_n\\ x\to i\infty}}$$

$$=\frac{\partial}{\partial\zeta}(\psi_1^1e^{-i\zeta x/\alpha_1})\Big|_{\substack{\zeta=-iK_n\\ x\to\infty}} \qquad \text{A}(2.18)$$

$\zeta=-iK_n$ 为下半平面的点。由 A(2.8)得

$$\psi^1(x-iK_n)e^{-K_nx/\alpha_1}=c_1\varphi^2(x,\ -iK_n)e^{-K_nx/\alpha_2}$$
$$+c_2\varphi^3(x,\ -iK_n)e^{-K_nx/\alpha_3}$$

故在 $-iK_n$ 点，$\psi^1(x,-iK_n)e^{-K_nx/\alpha_1}$ 是存在，并由等式右边表出。又设在 $-iK_n$ 的邻近 $\psi^1(x,\ \zeta)e^{-i\zeta x/\alpha_1}$ 也存在，代入(24)的第1式右端方括号内，并对等式两端取热限。便得

$$\lim_{x\to\infty}(\psi_1^1(x,\ \zeta)e^{-i\zeta x/\alpha_1})=1 \qquad \text{A}(2.19)$$

由 A(2.18)，A(2.19)便得

$$\lim_{x_2\to\infty}\lim_{\zeta\to-iK_n}\frac{A_1}{\zeta+iK_n}=\lim_{\zeta\to-iK_n}\lim_{x_2\to\infty}\frac{A_1}{\zeta+iK_n}=0 \qquad \text{A}(2.20)$$

由(51)、A(2.8)～A(2.20)诸式得

$$(T_{11}(\zeta)-1)\psi^1(x,\ \zeta)e^{-i\zeta y/\alpha_1}$$

在 $\zeta=-iK_n$ 点的留数 R 为

$$R=\lim_{\zeta\to-iK_n}(\zeta+iK_n)(T_{11}(\zeta)-1)\psi^1(x,\ \zeta)e^{-i\zeta x/\alpha_1}$$

$$=\lim_{\zeta\to-iK_n}\frac{\psi^1(x,\ \zeta)e^{-i\zeta x/\alpha_1}}{\dfrac{W}{\zeta+iK_n}}$$

$$=\frac{\begin{pmatrix}c_1\phi^2(x,\ -iK_n)e^{-K_nx/\alpha_1}+c_2\phi^3\\ (x,-iK_n)e^{-K_nx/\alpha_1}\end{pmatrix}}{\lim\limits_{\zeta\to-iK_n}\dfrac{A_2}{\zeta+iK_n}} \qquad \text{A}(2.21)$$

若 $c_2=0$，则 R 可写为

$$im_{2n}e^{-K_{2n}x/\alpha_1}\phi^2(x,\ -iK_{2n})$$

$$m_{2n}^{-1}=-\int_{-\infty}^{\infty}\det\left|\begin{matrix}v_1/\alpha_1 & v_1 & w_1\\ v_2/\alpha_2 & v_2 & w_2\\ v_3/\alpha_3 & v_3 & w_3\end{matrix}\right|e^{-K_{2n}y/\alpha'}dy \qquad \text{A}(2.22)$$

若 $c_1=0$，则 R 可写为

$$im_{3n}e^{-K_{3n}x/\alpha_1}\phi^3(x,\ -iK_n)$$

$$m_{3n}^{-1}=-\int_{-\infty}^{\infty}\det\left|\begin{matrix}w_1/\alpha_1 & v_1 & w_1\\ w_2/\alpha_2 & v_2 & w_2\\ w_3/\alpha_3 & v_3 & w_3\end{matrix}\right|e^{-K_{2n}y/\alpha'}dy \qquad \text{A}(2.23)$$

由 A(2.22)、A(2.23)便得(65)式。(65)式中对 n 求和，包括了各种留数。

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